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Uniform Asymptotic Expansions for Charlier Polynomials

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The asymptotic behaviour of the Charlier polynomials $C_n^{(a)}(x)$ as $n \to \infty$ is examined. These polynomials satisfy a discrete orthogonality relation and, unlike classical orthogonal polynomials, do not satisfy a second-order linear differential equation with respect to the independent variable x. As such, previous results on their asymptotic behaviour have been restricted to integral methods and consequently have been quite limited in their scope. In this paper a new approach is used, where the polynomials $C_n^{(a)}(x)$ are not regarded as a functions of x with a as a parameter, but rather with the roles reversed via a second-order linear differential equation in which a is the (real or complex-valued) independent variable and x is a parameter. This equation has two turning points in the a plane which depend on x, and are either positive or complex conjugates, according to the values of x. Moreover, the turning points can coalesce with one another, or one with a singularity of the equation, for certain critical values of x. By using two general asymptotic theories of differential equations, one for intervals free of turning points and the other for intervals containing a double pole and a coalescing turning point, expansions are derived for $C_n^{(a)}(x)$ involving either elementary functions or Bessel functions. Taken together, the results are uniformly valid for $-\infty < x < \infty$. In addition, in each case the expansions are uniformly valid for a lying in certain unbounded intervals, each of which contain $[-\{1-\delta\}\, n, \{1-\delta\}\, n]$, where $\delta \in (0, 1)$ is an arbitrary constant.

1. INTRODUCTION

In this paper we examine the asymptotic behaviour, as $n \to \infty$, of the Charlier polynomials, which are defined by

$$C_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} k!(-a)^{n-k}. \tag{1.1}$$

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The polynomials are generated by

$$e^{-aw}(1+w)^x = \sum_{n=0}^{\infty} C_n^{(a)}(x) \frac{w^n}{n!} \tag{1.2}$$

and are important because they satisfy the discrete orthogonality relation

$$\sum_{x=0}^{\infty} \omega(x)\, C_m^{(a)}(x)\, C_n^{(a)}(x) = a^n n!\, \delta_{mn} \qquad (a>0), \tag{1.3}$$

where

$$\omega(x) = \frac{e^{-a} a^x}{x!}. \tag{1.4}$$

We also remark that they satisfy the following recursion relation

$$C_{n+1}^{(a)}(x) = (x-n-a)\, C_n^{(a)}(x) - anC_{n-1}^{(a)}(x). \tag{1.5}$$

For further properties of these nonclassical polynomials, see [2] and [7].

We shall consider several cases as described below, which taken together will result in asymptotic approximations uniformly valid for $-\infty < x < \infty$. In addition, our results will, in aggregate, be valid for

$$-\{1-\delta\}\, n \leqslant a \leqslant \{1-\delta\}\, n, \tag{1.6}$$

where δ is an arbitrary constant satisfying $0<\delta<1$. In fact, for each case the a interval of validity will be larger than (1.6) and indeed will be allowed to be unbounded, either to $+\infty$ or to $-\infty$, according to the particular range of x under consideration.

The difficulty in obtaining useful asymptotic approximations for the Charlier, and other non-classical orthogonal polynomials (such as Meixner and Pollaczek), is that they do not satisfy a second-order linear differential equation with respect to the independent variable x. So far integral methods have been employed, with x generally being very restricted. Recently, for example, Goh [3] used an integral representation and obtained approximations of Plancherel–Rotach type, with rather weak error estimates. He considered seven x intervals, which taken together cover part (but certainly not all) of the interval

$$\varepsilon n \leqslant x \leqslant Mn \qquad (0<\varepsilon<M<\infty). \tag{1.7}$$

We also mention that Maejima and Van Assche [4] obtained some approximations for the case x negative.

Rui and Wong [6] improved the results of [3], using a contour integral representation and uniform asymptotics, which unified the results of [3] to one x interval. They employed a suitable transformation of integration variable, followed by integration by parts, to obtain an asymptotic expansion of the form

$$C_n^{(a)}(x) = n!\, e^{a+nq} E^{(n-x)/2} \left\{ J_{x-n}(2n\sqrt{E}\,) \sum_{k=0}^{p-1} \frac{a_k}{n^k} + \sqrt{E}\, J'_{x-n}(2n\sqrt{E}\,) \sum_{k=0}^{p-1} \frac{b_k}{n^k} + \varepsilon_p \right\}, \tag{1.8}$$

where $E = E(x/n, n)$ and $q = q(x/n, n)$ are solutions of a certain system of nonlinear equations. They then showed that as $n \to \infty$

$$\varepsilon_p = J_{x-n}(2n\sqrt{E}\,)\, O(n^{-p}) + \sqrt{E}\, J'_{x-n}(2n\sqrt{E}\,)\, O(n^{-p}), \tag{1.9}$$

for x satisfying (1.7), with $a > 0$ fixed.

We shall extend Rui and Wong's results considerably further. First, we consider three main cases, which taken together provide asymptotic expansions which are uniformly valid for $-\infty < x < \infty$, complete with explicit error bounds. Our approach is quite different to previous work, in that we show that $C_n^{(a)}(x)$ satisfies a differential equation in which a is the independent variable, with x appearing as a parameter (see (1.12) below). This allows us to employ powerful existing asymptotic results for differential equations. We shall obtain asymptotic expansions for numerically satisfactory solutions of the differential equation, which are uniformly valid for the parameter x lying in certain large intervals. As a by-product of our approach, the parameter a will not be restricted to being fixed, or indeed bounded. Furthermore, our method allows a to be complex, although for conciseness we restrict a to being real in our final results. The methods and results of this paper are intended for future studies on the asymptotic distribution of zeros, as well as similar investigations on the Meixner and Pollaczek polynomials.

It is worth emphasizing the importance of *uniform* asymptotics, as can be seen by comparing the results of [3] with [6]. In the present paper, uniform asymptotic approximations are obtained which are valid in full neighborhoods of the singularities and the turning point, and also for the cases where a turning point is permitted to be arbitrarily close to, or indeed coincide with, a pole. This allows a, and more importantly x, to range over large intervals. Since x is a parameter in the equations, alternative (non-uniform) asymptotic methods can only yield a collection of weaker approximations for various (essentially fixed) values of x.

Generally, it is understood that the price paid for employing uniform asymptotics is the requirement of fairly complicated transformations, as well as the use of higher function approximants (in the present case Bessel functions, although some of our approximations only involve elementary functions). However, from the comments above we claim that in the study of non-classical polynomials the benefits of employing uniform asymptotic methods far outweigh this. Moreover, Bessel functions (especially of real variables) are now fairly easy to compute, and the transformed variables given below are either explicitly given, or are readily computable via an implicit relation.

To derive the appropriate differential equation, we make use of the following relationship with the Laguerre polynomials, or equivalently the confluent hypergeometric function:

$$C_n^{(a)}(x) = n! L_n^{(x-n)}(a) = \Gamma(x+1)\,\mathbf{M}(-n, x-n+1, a). \tag{1.10}$$

Here (using the notation of [5, Chap. 7, Sect. 9])

$$\mathbf{M}(-n, x-n+1, a) = \sum_{s=0}^{n} \frac{(-n)_s}{\Gamma(x-n+1+s)} \frac{a^s}{s!}. \tag{1.11}$$

From the confluent hypergeometric equation we then see that $C_n^{(a)}(x)$ satisfies the second-order linear differential equation (in the parameter a)

$$a\frac{d^2y}{da^2} + (1+x-n-a)\frac{dy}{da} + ny = 0. \tag{1.12}$$

We shall obtain our results via this equation with the first derivative removed. Thus, on setting

$$y(a) = e^{a/2} a^{(n-x-1)/2} w(a) \tag{1.13}$$

in (1.12) we obtain our desired form

$$\frac{d^2w}{da^2} = \left\{ \frac{(x-n)^2-1}{4a^2} - \frac{n+x+1}{2a} + \frac{1}{4} \right\} w. \tag{1.14}$$

Equations (1.12) and (1.14) have a regular singularity at $a=0$, and an irregular singularity at $a=\infty$. Our first task is to examine the behaviour of $C_n^{(a)}(x)$ at these singularities, and to introduce other solutions of (1.12) which are recessive at the singularities.

When $x \geqslant n$ the Charlier polynomial $C_n^{(a)}(x)$ is the recessive solution of (1.12) at $a=0$: as $a \to 0$ (with $x \geqslant n$)

$$C_n^{(a)}(x) \to \frac{\Gamma(x+1)}{\Gamma(x-n+1)}, \tag{1.15}$$

whereas all other independent solutions are unbounded as $a \to 0$ $(x \geqslant n)$.

When $x<n$ this is not always so: in this case the recessive solution at $a=0$ is given by

$$\mathbf{N}(-n,x-n+1,a)=a^{n-x}\mathbf{M}(-x,n-x+1,a). \tag{1.16}$$

We have as $a\to 0$

$$\mathbf{N}(-n,x-n+1,a)\sim\frac{a^{n-x}}{\Gamma(n-x+1)}, \tag{1.17}$$

so that $\mathbf{N}(-n,x-n+1,a)\to 0$ (for $x<n$). However, when $0\leqslant x<n$ and x is an integer, then (1.15) no longer holds, but instead as $a\to 0$

$$C_n^{(a)}(x)\sim\frac{n!}{(n-x)!}(-a)^{n-x}, \tag{1.18}$$

with (1.17) still holding. Hence $C_n^{(a)}(x)$ and $\mathbf{N}(-n,x-n+1,a)$ are linearly dependent in this case, namely

$$C_n^{(a)}(x)=n!(-1)^{n-x}\,\mathbf{N}(-n,x-n+1,a). \tag{1.19}$$

In summary, $C_n^{(a)}(x)$ is the recessive solution of (1.12) at $a=0$ when $x\geqslant n$, or when x is an integer with $0\leqslant x\leqslant n$. For all other values of x, $C_n^{(a)}(x)$ is a dominant solution at $a=0$: in this Case (1.17) still holds, whereas

$$C_n^{(a)}(x)\to(-1)^n\frac{\Gamma(n-x)}{\Gamma(-x)} \tag{1.20}$$

as $a\to 0$ (for fixed $x\in(-\infty,0)$, or fixed non-integer $x\in(0,n)$).

For all values of x we can also use the fact that $C_n^{(a)}(x)$ is the recessive solution at $a=+\infty$: from (1.1)

$$C_n^{(a)}(x)\sim(-a)^n \tag{1.21}$$

as $a\to\infty$, whereas all other independent solutions of (1.12) are exponentially large in a as $a\to+\infty$ (for each fixed x in the interval $-\infty<x<\infty$).

The solution of (1.12) which is recessive at $a=-\infty$ ($\arg(a)=\pi$) is given by

$$V(-n,x-n+1,a)=e^aU(x+1,x-n+1,ae^{-\pi i}), \tag{1.22}$$

where, for $|\arg(z)| < \frac{1}{2}\pi$ and $\operatorname{Re} p > 0$,

$$U(p, q, z) = \frac{1}{\Gamma(p)} \int_0^\infty t^{p-1}(1+t)^{q-p-1} e^{-zt}\, dt. \tag{1.23}$$

As $a \to -\infty$

$$V(-n, x-n+1, a) \sim e^a(-a)^{-x-1}, \tag{1.24}$$

and so is exponentially small (compare (1.21)). Also, as $a \to 0$ with $x < n$

$$V(-n, x-n+1, a) \to \frac{\Gamma(n-x)}{n!}. \tag{1.25}$$

Unlike $\mathbf{N}(-n, x-n+1, a)$, the solution $V(-n, x-n+1, a)$ is independent of $C_n^{(a)}(x)$ (as a function of a) for all fixed values of x.

We shall obtain asymptotic expansions for $\mathbf{N}(-n, x-n+1, a)$ and $V(-n, x-n+1, a)$ by matching them with asymptotic solutions of (1.14) which are recessive at $a = 0$ and $a = -\infty$, respectively. Similarly, for certain values of x, we shall obtain asymptotic expansions for $C_n^{(a)}(x)$ directly by matching it with asymptotic solutions of (1.14) which are recessive at $a = 0$, or $a = +\infty$. For the remaining values of x we shall employ our asymptotic expansions for $\mathbf{N}(-n, x-n+1, a)$ and $V(-n, x-n+1, a)$ and appeal to the connection formula

(1.26)

$$C_n^{(a)}(x) = n!\, e^{-(n-x)\pi i}\mathbf{N}(-n, x-n+1, a) + \frac{n!(-1)^n}{\Gamma(-x)} V(-n, x-n+1, a).$$

With these considerations in mind, we shall consider three cases separately. In order to describe these, we introduce the following parameters. Let

$$u = n + \tfrac{1}{2}, \tag{1.27}$$

$$\alpha(x) = \frac{x-n}{u}, \tag{1.28}$$

$$\beta(x) = -\frac{2x+1}{n-x}, \tag{1.29}$$

$$s^-(\beta) = 1 - \beta - \sqrt{\beta^2 - 2\beta}\,, \tag{1.30}$$

$$t^+(\alpha) = 2 + \alpha + 2\sqrt{1+\alpha}\,, \tag{1.31}$$

and

$$v(x)=\sqrt{\frac{u}{x+\frac{1}{2}}}. \tag{1.32}$$

In addition, for $0 \leqslant \beta \leqslant 1$ define $s_0(\beta) \in (0, 1]$ implicitly by

$$S_0-\beta \ln(1-\beta-s_0+S_0)+\ln\left(\frac{s_0^2+\beta s_0+1-(s_0+1)\, S_0}{s_0}\right) = \frac{1}{2}(2-\beta)\ln(2-\beta)-\frac{1}{2}\beta\ln(\beta), \tag{1.33}$$

where

$$S_0=\sqrt{s_0^2-2s_0+2\beta s_0+1}. \tag{1.34}$$

(It is understood that $s_0(0)=\lim_{\beta\to 0} s_0(\beta)=1$.) In the appendix we prove that $s_0(\beta)$ is monotonically decreasing for $0 \leqslant \beta \leqslant 1$. From this it follows that

$$s_0(1) \leqslant s_0(\beta) \leqslant s_0(0)=1, \tag{1.35}$$

where $s_0(1)=0.6627...$.

With these definitions, we illustrate the three cases in Table 1.

Here Δ is an arbitrary constant satisfying $3 \leqslant \Delta < \infty$, and we use δ generically as an arbitrary constant satisfying $0<\delta<1$. From (1.7) we observe that Rui and Wong [6] considered Case II (with a fixed).

We will show later that the a interval of validity of Case Ia is larger than $-\{1-\delta\}\, u \leqslant a < \infty$, and the a intervals of validity of Cases Ib, Ic, II and III are all larger than $-\infty < a \leqslant \{1-\delta\}\, u$.

In Case I, which covers the x range $-\infty < x \leqslant \delta^4 u - \frac{1}{2}$, the approximants are elementary (exponential) functions. There are two turning points involved (in the complex a plane), which restrict the region of validity of the independent variable a. In Case Ia ($1<\beta<2$) the turning points are complex conjugates and lie in the left half plane, and asymptotic expansions are derived directly for $C_n^{(a)}(x)$ via a matching at $a=+\infty$ (where it is recessive). In Cases Ib ($0<\beta \leqslant 1$) and Ic the turning points lie in the right half plane (complex conjugates for Case Ib, and real and positive for Case Ic), and this prevents us from obtaining asymptotic expansions directly for $C_n^{(a)}(x)$ via a matching at $a=+\infty$. Therefore asymptotic expansions are instead derived directly for $\mathbf{N}(-n, x-n+1, a)$ and $V(-n, x-n+1, a)$, with the corresponding compound expansion for $C_n^{(a)}(x)$ coming from the connection formula (1.26). Case Ic is very similar

TABLE 1

Case	Asymptotic parameter	$\alpha(x)$ interval	**Subcase**	x interval	a interval
I	$n-x$	$-\infty < \alpha(x) \leqslant -1+\delta^4$			
			a	$-\infty < x < -n-1$	$-\{1-\delta\}\, s_0(2-\beta)(n-x) \leqslant a < \infty$
			b	$-n-1 \leqslant x < -\frac{1}{2}$	$-\infty < a \leqslant \{1-\delta\}\, s_0(\beta)(n-x)$
			c	$-\frac{1}{2} \leqslant x \leqslant \delta^4 u - \frac{1}{2}$	$-\infty < a \leqslant \{1-\delta\}\, s^{-}(\beta)(n-x)$
II	u	$-1+\delta^4 \leqslant \alpha(x) \leqslant \Delta$			
			a	$\delta^4 u - \frac{1}{2} \leqslant x < n$	$-\infty < a \leqslant \{t^{+}(\alpha)-\delta\}\, u$
			b	$n \leqslant x \leqslant (1+\Delta)\, n + \frac{1}{2}\Delta$	$-\infty < a \leqslant \{t^{+}(\alpha)-\delta\}\, u$
III	$x+\frac{1}{2}$	$\Delta \leqslant \alpha(x) < \infty$		$(1+\Delta)\, n + \frac{1}{2}\Delta \leqslant x < \infty$	$-\infty < a \leqslant \{1-\delta\}(1-v(x))^2\,(x+\frac{1}{2})$

to Case Ib, except that the two turning points are real as opposed to complex, and subsequently the semi-infinite interval of validity of the variable a differs slightly.

In Case II, which covers the x range $\delta^4 u - \frac{1}{2} \leqslant x \leqslant (1+\Delta)\, n + \frac{1}{2}\Delta$, the approximants are Bessel functions (for $a > 0$) or modified Bessel functions (for $a < 0$). In this case the region of validity with respect to the variable a includes a turning point and a double pole, and furthermore these two points can coalesce for certain critical values of x. In Case IIa we see from Table 1 that $x < n$ and $a = +\infty$ is not included in the interval of validity, and as such $\mathbf{N}(-n, x-n+1, a)$ and $V(-n, x-n+1, a)$ must be used as the numerically satisfactory solutions, which can be identified directly with the asymptotic solutions which are recessive at $a = 0$ and $a = -\infty$, respectively. As in Cases Ib and Ic, the corresponding compound expansion for $C_n^{(a)}(x)$ then comes from the connection formula (1.26). In Case IIb we see that $x \geqslant n$, which means that asymptotic expansions can be derived directly for $C_n^{(a)}(x)$ via a matching of recessive solutions at $a = 0$.

In Case III, which covers the x range $(1+\Delta)\, n + \frac{1}{2}\Delta \leqslant x < \infty$, the approximants are elementary (exponential) functions. There are two real turning points, but they are bounded away from the interval $-\infty < a \leqslant \{1-\delta\}(1-v(x))^2 (x+\frac{1}{2})$. Since $x > n$ in this case, we are able to obtain an asymptotic expansion for $C_n^{(a)}(x)$ from a direct matching of recessive solutions at $a = 0$.

Case I is covered in Sections 2 and 3, Case II in Sections 4 and 5, and Case III in Sections 6 and 7. The main results are summarised in Section 8, and some numerical examples are given in Section 9.

2. CASE I: PRELIMINARY TRANSFORMATIONS

In this case $-\infty < \alpha(x) \leqslant -1 + \delta^4$, which is equivalent to $-\infty < x \leqslant \delta^4 u - \frac{1}{2}$ (recall that $u = n + \frac{1}{2}$). We shall take $n - x$ as the large asymptotic parameter, and rescale the independent variable in (1.14) by

$$s = \frac{a}{n-x}. \tag{2.1}$$

As a result we arrive at the differential equation

$$\frac{d^2 w}{ds^2} = \{(n-x)^2 \tilde{f}(s) + \tilde{g}(s)\}\, w, \tag{2.2}$$

where

$$\tilde{f}(s) = \frac{s^2 - 2s + 2\beta s + 1}{4s^2}, \tag{2.3}$$

and

$$\tilde{g}(s) = -\frac{1}{4s^2}. \tag{2.4}$$

Recall that $\beta(x)$ is defined by (1.29). Now, for fixed n we see that $\beta(x)$ is a decreasing function of x in the range $-\infty < x < n$, and therefore for $-\infty < x \leqslant \delta^4 u - \frac{1}{2} < n$ we deduce that

$$-\frac{2\delta^4}{1-\delta^4} \leqslant \beta(x) < 2. \tag{2.5}$$

For large $n-x$ the zeros of $\tilde{f}(s)$ are turning points of the differential Eq. (2.2). From (2.3) we observe that there are two turning points, at $s = s^{\pm}(\beta)$, say. When $0 < \beta < 2$ these are complex conjugates, located on the unit circle $|s| = 1$ at

$$s^{\pm}(\beta) = 1 - \beta \pm i\sqrt{\beta(2-\beta)}\,. \tag{2.6}$$

When $\beta \leqslant 0$ they are real and positive, located at

$$s^{\pm}(\beta) = 1 - \beta \pm \sqrt{\beta^2 - 2\beta}\,. \tag{2.7}$$

They are also real for $\beta \geqslant 2$, but from (2.5) we see that this range will not being considered here.

The two turning points coalesce at $s = 1$ when $\beta = 0$ (i.e., $x = -\frac{1}{2}$), and they coalesce at $s = -1$ when $\beta \to 2$ (i.e., $x \to -\infty$). On account of the position of these turning points in the s plane as β varies, we shall consider the following three subcases.

- In Subcase Ia we consider the turning points lying on the unit semi-circle $|s| = 1$ in the left-half s plane. This occurs for $1 < \beta < 2$, or equivalently $-\infty < x < -n-1$. We shall show below that the position of the turning points results in the asymptotic approximations being valid for $-\{1-\delta\}\, s_0(2-\beta) \leqslant s < \infty$ (i.e., $-\{1-\delta\}\, s_0(2-\beta)(n-x) \leqslant a < \infty$).
- In Subcase Ib the turning points lie on the unit semi-circle $|s| = 1$ in the right-half s plane. This occurs for $0 < \beta \leqslant 1$ (i.e., $-n-1 \leqslant x < -\frac{1}{2}$). As a result the subsequent asymptotic expansions will be valid for $-\infty < s \leqslant \{1-\delta\}\, s_0(\beta)$ (i.e., $-\infty < a \leqslant \{1-\delta\}\, s_0(\beta)(n-x)$).
- In Subcase Ic the turning points are real and positive, located at (2.7). This occurs for $-2\delta^4/(1-\delta^4) \leqslant \beta \leqslant 0$ (i.e., $-\frac{1}{2} \leqslant x \leqslant \delta^4 u - \frac{1}{2}$). The asymptotic expansions this time will be valid for $-\infty < s \leqslant \{1-\delta\}\, s^-(\beta)$: this ensures that s is bounded away from both real turning points.

Before proceeding with the required transformations, we show that the a interval of validity of Case Ia is larger than $-\{1-\delta\}\, u \leqslant a < \infty$, and the a intervals of validity of Cases Ib and Ic are both larger than $-\infty \leqslant a\{1-\delta\}\, u$.

Consider first Case Ib: numerical calculations indicate that $s_0(\beta) \geqslant 1-\frac{1}{2}\beta$ for $0 \leqslant \beta \leqslant 1$: see Fig. 1.

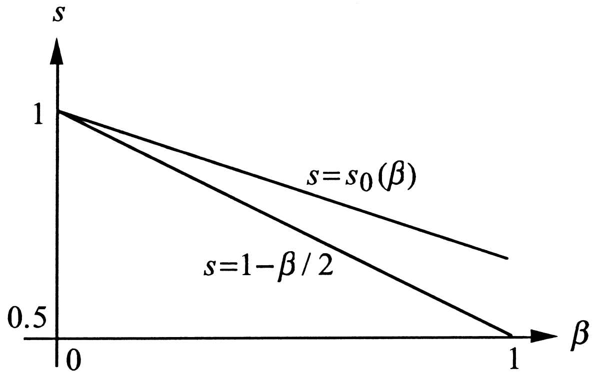

FIGURE 1

Now from (1.29) we see that $s_0(\beta) \geqslant 1-\frac{1}{2}\beta$ is equivalent to $s_0(\beta) \geqslant u/(n-x)$. Therefore, for each fixed x under consideration, the a-interval of validity $-\infty < a \leqslant \{1-\delta\}\, s_0(\beta)(n-x)$ in Case Ib is larger than $-\infty \leqslant a < \{1-\delta\}\, u$. Similarly, for Case Ia (in which $1 < \beta < 2$), we replace β by $2-\beta$ in the inequality $s_0(\beta) \geqslant 1-\frac{1}{2}\beta$, and this gives

$$s_0(2-\beta) \geqslant \frac{1}{2}\beta = -\frac{x+\frac{1}{2}}{n-x} > \frac{u}{n-x}, \tag{2.8}$$

since $x < -n-1$ in this case: consequently, we see that the a-interval of validity $-\{1-\delta\}\, s_0(2-\beta)(n-x) \leqslant a < \infty$ for Case Ia is larger than $-\{1-\delta\}\, u \leqslant a < \infty$.

Finally, for Case Ic, we first observe that $s^-(\beta)$ is increasing for $-\infty < \beta < 0$, since

$$\frac{\mathrm{d}s^-(\beta)}{\mathrm{d}\beta} = \frac{1-\beta-\sqrt{\beta^2-2\beta}}{\sqrt{\beta^2-2\beta}} > \frac{1-\beta-\sqrt{\beta^2-2\beta+1}}{\sqrt{\beta^2-2\beta}} = 0. \tag{2.9}$$

Therefore, using the inequality $-2\delta^4/(1-\delta^4) \leqslant \beta \leqslant 0$ and (2.7), we find that the smaller turning point lies in the interval

$$\frac{1-\delta^2}{1+\delta^2} \leqslant s^-(\beta) \leqslant 1. \tag{2.10}$$

On the other hand, since $s=a/(n-x)$ and $-\infty<x\leqslant\delta^4u-\frac{1}{2}$, we see that the condition $-\infty<a\leqslant\{1-\delta\}\,u$ implies that

$$-\infty<s\leqslant\frac{\{1-\delta\}\,u}{n-x}\leqslant\frac{1}{(1+\delta)(1+\delta^2)}. \tag{2.11}$$

Therefore from (2.10) and (2.11)

$$s^-(\beta)-s\geqslant\frac{\delta(1-\delta-\delta^2)}{(1+\delta)(1+\delta^2)}>0, \tag{2.12}$$

provided that $\delta>0$ and $1-\delta-\delta^2>0$, or equivalently $0<\delta<\frac{1}{2}(\sqrt{5}-1)$. Thus $a\in(-\infty,\{1-\delta\}\,u]$ is sufficient to ensure that $s\in(-\infty,\{1-\delta\}\,s^-(\beta)]$.

Let us return to Eq. (2.2). In order to obtain asymptotic solutions, we make the Liouville transformation

$$\xi=\int\sqrt{\tilde{f}(s)}\,\mathrm{d}s=\int\frac{\sqrt{s^2-2s+2\beta s+1}}{2s}\,\mathrm{d}s, \tag{2.13}$$

and

$$\tilde{W}=\tilde{f}^{1/4}(s)\,w. \tag{2.14}$$

See [5, Chap. 10] for details. The resulting equation takes the form

$$\frac{\mathrm{d}^2\tilde{W}}{\mathrm{d}\xi^2}=\{(n-x)^2+\tilde{\psi}(\xi)\}\,\tilde{W}, \tag{2.15}$$

where

$$\tilde{\psi}(\xi)=\frac{\tilde{g}(s)}{\tilde{f}(s)}+\frac{4\tilde{f}(s)\,\tilde{f}''(s)-5\tilde{f}'^2(s)}{16\tilde{f}^3(s)}. \tag{2.16}$$

Using (2.3) and (2.4) we have explicitly

$$\tilde{\psi}(\xi)=\frac{10(1-\beta)\,s-(15-14\beta+7\beta^2)\,s^2+8(1-\beta)\,s^3-s^4-2}{(s^2-2s+2\beta s+1)^3}. \tag{2.17}$$

We shall define the branch of the s-ξ transformation according to the value of β. For Cases Ib and Ic we integrate (2.13) to yield $\xi=\xi_b$, where

$$\xi_b=\frac{1}{2}\sqrt{s^2-2s+2\beta s+1}-\frac{1}{2}\beta\ln(1-\beta-s+\sqrt{s^2-2s+2\beta s+1}\,)+\frac{1}{2}\ln\left(\frac{(s+1)\sqrt{s^2-2s+2\beta s+1}-s^2-\beta s-1}{s}\right). \tag{2.18}$$

Here and throughout the square roots are taken to be positive for real s. The branches for the logarithms are chosen so that ξ_b is real for $\arg(s)=\pi\ (-\infty<s<0)$.

For Case Ia $(1<\beta<2)$, we prefer ξ to be real for $0<s<\infty$ $(\arg(s)=0)$, and so take the arbitrary integration constant in (2.13) with this in mind. Thus explicit integration gives $\beta=\xi_a$, where

$$\xi_a=\frac{1}{2}\sqrt{s^2-2s+2\beta s+1}-\frac{1}{2}\beta\ln(1-\beta-s+\sqrt{s^2-2+2\beta s+1}\,)+\frac{1}{2}\ln\left(\frac{s^2+\beta s+1-(s+1)\sqrt{s^2-2s+2\beta s+1}}{s}\right). \tag{2.19}$$

Thus for $-\infty<s<0$ $(\arg(s)=\pi)$ we see that $\operatorname{Im}\xi_a=\frac{1}{2}\pi$. Note that $\xi_a=\xi_b+\frac{1}{2}\pi i$ for real s $(\arg(s)=0$ or $\arg(s)=\pi)$.

When $0<\beta<2$ the turning point in the upper half-plane $s=s^+(\beta)=1-\beta+i\sqrt{\beta(2-\beta)}$ is mapped to

$$\xi_a^+(\beta)=\tfrac{1}{4}(2-\beta)\ln(2-\beta)-\tfrac{1}{4}\beta\ln(\beta)+\tfrac{1}{4}\beta\pi i \qquad (1<\beta<2), \tag{2.20}$$

and

$$\xi_b^+(\beta)=\tfrac{1}{4}(2-\beta)\ln(2-\beta)-\tfrac{1}{4}\beta\ln(\beta)-\tfrac{1}{4}(2-\beta)\pi i \qquad (0\leqslant\beta\leqslant 1). \tag{2.21}$$

When $\beta\leqslant 0$ the (real) turning points at $s=s^\pm(\beta)=1-\beta\pm\sqrt{\beta^2-2\beta}$ are mapped to

$$\xi_b^-(\beta)=\tfrac{1}{2}\ln(2-\beta)-\tfrac{1}{4}\beta\ln(\beta^2-2\beta)-\tfrac{1}{2}\pi i, \tag{2.22}$$

and

$$\xi_b^+(\beta)=\tfrac{1}{2}\ln(2-\beta)-\tfrac{1}{4}\beta\ln(\beta^2-2\beta)-\tfrac{1}{2}(1-\beta)\pi i, \tag{2.23}$$

respectively. Note that (2.20)–(2.23) are branch points of the s-ξ transformation.

Although we are primarily concerned with real values of s (equivalently a), on account of the branch point of the transformations at $s=0$ we must consider complex values of ξ. In particular, we consider $0 \leqslant \arg(s) \leqslant \pi$, and for (2.18) introduce a branch cut parallel to the imaginary axis from $\xi_b = \xi_b^+(\beta)$ to $\xi_b = \operatorname{Re} \xi_b^+(\beta) - i\infty$: then the branch of the right hand side of (2.18) is defined as being real for $\arg(s) = \pi$ and by continuity elsewhere in the cut ξ_b plane. The map of the upper half s plane ($0 \leqslant \arg(s) \leqslant \pi$) to the ξ_b plane is indicated in Fig. 2a, and 2b, with corresponding points indicated by the same capital letters. In Fig. 2a the ray **DB**′ emanating from the branch point at $\xi = \xi_b^+(\beta)$, parallel to the imaginary ξ_b axis, is important as it is a boundary for region of validity of the subsequent asymptotic expansions. In the s plane the corresponding curve intersects the positive real axis at the point labeled **B**′. This occurs at $s_0(\beta)$, where $s = s_0(\beta)$ is the real solution of (2.18) when $\xi_b = \operatorname{Re} \xi_b^+(\beta) - \frac{1}{2}\pi i$. This is easily shown to be equivalent to the earlier definition (1.33) of $s_0(\beta)$. One can then show from (2.18) and the definition (1.33) that the point $s = -s_0(2-\beta)$ on the negative real axis ($\arg(s) = \pi$) corresponds to $\xi_b = \operatorname{Re} \xi_b^+(\beta)$: this point is labelled **B** in Figs. 2a and 2b.

Likewise, for the transformation (2.19) when $0 \leqslant \arg(s) \leqslant \pi$, we introduce a branch cut parallel to the imaginary axis from $\xi_a = \xi_a^+(\beta)$ to $\xi_a = \xi_a^+(\beta) + i\infty$, and take the branch of the right hand side of (2.19) to be real for $\arg(s) = 0$, and by continuity elsewhere in the cut ξ_a plane. The Scharzian derivative $\tilde{\psi}(\xi)$ is analytic in the cut ξ_a and ξ_b planes.

We next require the behaviour of ξ as s approaches the singularities of the differential Eq. (2.2). Firstly, from (2.18) and (2.19) we find that

$$\xi_{a,b} = \tfrac{1}{2} \ln(\tfrac{1}{2}|s|) + \tfrac{1}{2}(2-\beta)\ln(2-\beta) + \tfrac{1}{2} + O(s) \tag{2.24}$$

as $s \to 0+$ (Case Ia) or $s \to 0-$ (Cases Ib, c). Next, as $s \to \infty$ (Case Ia) it is straightforward to show that

$$\xi_a = \tfrac{1}{2} s + \tfrac{1}{2}(\beta - 1)\ln(2es) - \tfrac{1}{2}\beta \ln(\beta) + \tfrac{1}{2}(2-\beta)\ln(2-\beta) + O(s^{-1}), \tag{2.25}$$

and as $s \to -\infty$ (Cases Ib, c)

$$\xi_b = \tfrac{1}{2}|s| + \tfrac{1}{2}(1-\beta)\ln(2e|s|) + O(s^{-1}). \tag{2.26}$$

3. CASE I: ASYMPTOTIC EXPANSIONS

We now apply Theorem 3.1 of [5, Chap. 10] (with $n-x$ playing the role of u in Olver's Theorem) to Eq. (2.15). As a result we obtain the following asymptotic solution which is recessive at $\xi = -\infty$ ($s = 0$)

$$\tilde{W}_{N,1}(n, x, \xi) = e^{(n-x)\xi}\left[1 + \sum_{j=1}^{N-1} \frac{\tilde{A}_j(\xi)}{(n-x)^j}\right] + \tilde{\varepsilon}_{N,1}(n, x, \xi). \tag{3.1}$$

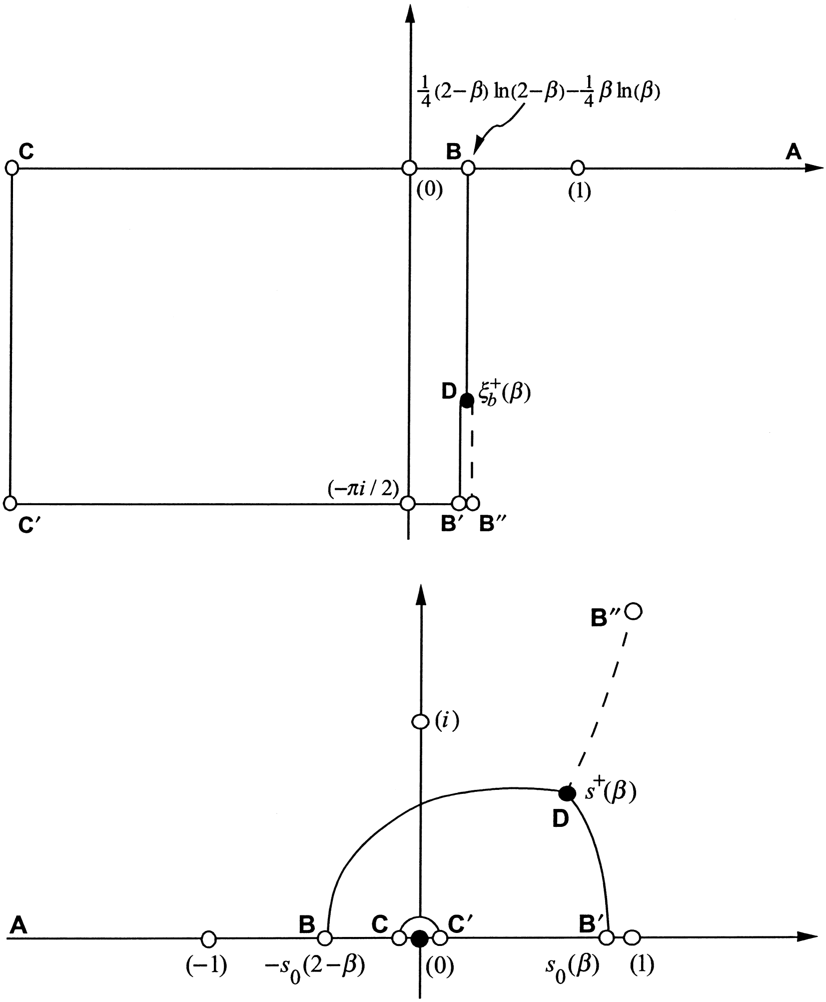

FIG. 2. (a) ξ_b plane. (b) s plane $(0 \leqslant \arg(s) \leqslant \pi)$.

Here $\tilde{A}_0(\xi) = 1$ and

$$\tilde{A}_{j+1}(\xi) = -\tfrac{1}{2}\,\tilde{A}_j'(\xi) + \tfrac{1}{2}\int^{\xi} \tilde{\psi}(\xi)\,\tilde{A}_j(\xi)\,\mathrm{d}\xi + \lambda_j \qquad (j = 0, 1, 2, \ldots). \tag{3.2}$$

We choose the integration constants λ_j such that

$$\lim_{\substack{\xi \to -\infty \\ (s \to 0)}} \tilde{A}_j(\xi) = 0 \qquad (j = 1, 2, 3, \ldots). \tag{3.3}$$

In (3.1) we allow both $\xi=\xi_a$ and $\xi=\xi_b$. The error term satisfies the following bound

$$|\tilde{\varepsilon}_{N,1}(n,x,\xi)| \leqslant 2e^{(n-x)\xi} \exp\left\{\frac{2\mathscr{V}_{-\infty,\xi}(\tilde{A}_1)}{n-x}\right\} \frac{\mathscr{V}_{-\infty,\xi}(\tilde{A}_N)}{(n-x)^N}, \tag{3.4}$$

for certain values of ξ, as described later. The variational operator $\mathscr{V}$ is defined as in [5, Chap. 1, Sect. 11]. Note that $e^{-(n-x)\xi}\tilde{\varepsilon}_{N,1}(n,x,\xi)\to 0$ as $s\to 0$ ($\xi\to-\infty$). We remark that all regions of validity are unbounded, on account of the fact that $\tilde{\psi}(\xi)=O(\xi^{-2})$ as $\xi\to\infty$: see (2.17), (2.24)–(2.26), and [5, Chap. 10, Ex. 5.1].

Two more asymptotic solutions, which are recessive at $s=\pm\infty$ ($\xi_a=\infty$ and $\xi_b=\infty$, respectively), are furnished by

$$\tilde{W}^{\pm}_{N,2}(n,x,\xi)=e^{-(n-x)\xi}\left[1+\sum_{j=1}^{N-1}(-1)^j\frac{\tilde{A}_j(\xi)}{(n-x)^j}\right]+\tilde{\varepsilon}^{\pm}_{N,2}(n,x,\xi). \tag{3.5}$$

This time the error terms are bounded by

$$|\tilde{\varepsilon}^{\pm}_{N,2}(n,x,\xi)| \leqslant 2e^{-(n-x)\xi} \exp\left\{\frac{2\mathscr{V}_{\xi,\infty}(\tilde{A}_1)}{n-x}\right\} \frac{\mathscr{V}_{\xi,\infty}(\tilde{A}_N)}{(n-x)^N}, \tag{3.6}$$

where $\xi=\xi_a$ for the plus superscript, and $\xi=\xi_b$ for the minus superscript. Note that for the plus superscript $\xi_a=+\infty$ corresponds to $s=+\infty$, and for the minus superscript $\xi_b=+\infty$ corresponds to $s=-\infty$. Thus $e^{(n-x)\xi_a}\tilde{\varepsilon}^{+}_{N,2}(n,x,\xi_a)\to 0$ as $s\to+\infty$, whereas $e^{(n-x)\xi_b}\tilde{\varepsilon}^{-}_{N,2}(n,x,\xi_b)\to 0$ as $s\to-\infty$. We emphasize that $\tilde{W}^{+}_{N,2}(n,x,\xi_a)$ and $\tilde{W}^{-}_{N,2}(n,x,\xi_b)$ are linearly independent solutions of (2.15) for all values of x and n.

When ξ is real the variation paths in (3.4) and (3.6) are taken along the real axis. However, when ξ is complex the variation paths must be chosen so that as a point v (say) passes along the path from $\pm\infty$ to ξ, $\operatorname{Re} v$ must be nondecreasing (for $\tilde{\varepsilon}_{N,1}(n,x,\xi)$) or nonincreasing (for $\tilde{\varepsilon}^{\pm}_{N,2}(n,x,\xi)$). For the bound (3.4) on $\tilde{\varepsilon}_{N,1}(n,x,\xi)$, the variation path from $-\infty$ to all ξ (corresponding to any $s\in(-\infty,\infty)$) meeting the monotonicity requirements is always possible, provided that the turning points $s=s^{\pm}(\beta)$ are bounded away from the real axis. However, if the turning points are close to the real axis (at $s=-1$ in Case Ia, or at $s=1$ in Case Ib) the error

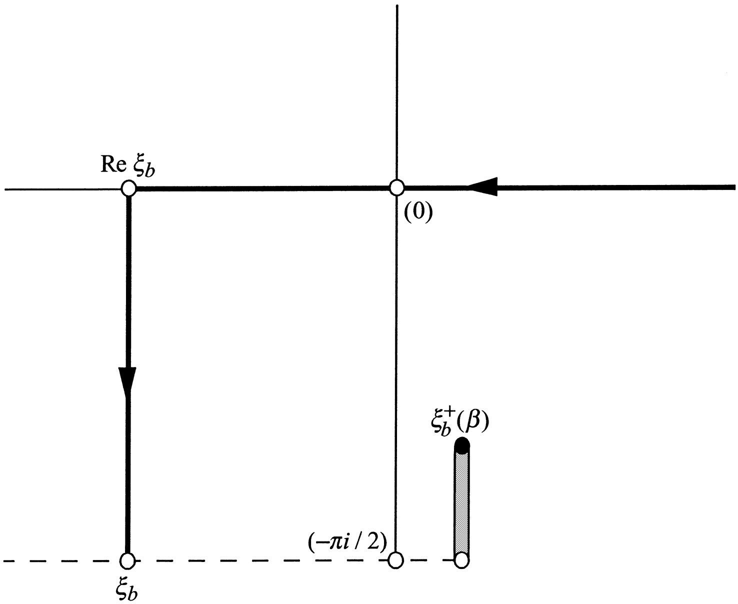

FIG. 3. Variation path.

bound (3.4) is restricted to a semi-infinite interval. Thus for Case Ia the bound (3.4) is valid for (at least) $s \in [-1+\delta, \infty)$, and for Case Ib the bound (3.4) is uniformly valid for (at least) $s \in (-\infty, 1-\delta]$.

Consider now the bound (3.6) when ξ is complex and s is real (so that $\operatorname{Im} \xi = \pm\frac{1}{2}\pi$). The simplest variation paths meeting the monotonicity requirements consist of the union of two lines, the first along the real axis from $+\infty$ to $\operatorname{Re} \xi$, and the second parallel to the imaginary axis from $\operatorname{Re} \xi$ to ξ. As an illustration (Case Ib) of a variation path from $+\infty$ to ξ_b, see Fig. 3. In this figure $s > 0$, and consequently ξ_b is complex with $\operatorname{Im} \xi_b = -\frac{1}{2}\pi$.

For (3.6) the variation path, from $+\infty$ to ξ (corresponding to a real value of s) is only possible for restricted values of ξ. Consider first $\tilde{\varepsilon}_{N,2}^{-}(n, x, \xi_b)$. From Fig. 2a we see that the restriction on the location of ξ_b is $-\infty < \operatorname{Re} \xi_b \leqslant (1-\delta) \operatorname{Re} \xi_b^+(\beta)$, on account of the branch point at $\xi = \xi_b^+(\beta)$. This leads to the restriction, for Case Ib (minus superscript), that $-\infty < s \leqslant \{1-\delta\}\, s_0(\beta)$ (equivalently $-\infty < a \leqslant \{1-\delta\}\, s_0(\beta)(n-x)$) for the bound (3.6) to hold if s is real. Similarly, for Case Ia (plus superscript), the bound (3.6) holds for $-\{1-\delta\}\, s_0(2-\beta) \leqslant s < \infty$. As discussed above, in Case Ic the bounds (3.4) and (3.6) both hold uniformly for $-\infty < s \leqslant \{1-\delta\}\, s^-(\beta)$.

Subcase Ia. We now are in a position to derive asymptotic expansions for the Charlier polynomials, as well as their companion solutions. First we make the identification

$$(3.7) \qquad C_n^{(a)}(x) = \tilde{K}_{N,2}^{+}(n, x)\, e^{a/2} a^{(n-x)/2} \times \{a^2 - 2(n+x+1)\, a + (n-x)^2\}^{-1/4}\, \tilde{W}_{N,2}^{+}(n, x, \xi_a),$$

where $\tilde{K}_{N,2}^{+}(n, x)$ is independent of s (or a). This relationship holds since both sides of (3.7) are solutions of (1.12) (see (1.13) and (2.14)) which are recessive at $a = \infty$. The bound (3.6) on the asymptotic solution $\tilde{W}_{N,2}^{+}(n, x, \xi_a)$ is uniformly valid for $-\{1-\delta\}\, s_0(2-\beta) \leqslant s < \infty$. This allows us to compare both sides as $a \to \infty$ (equivalently $\xi_a \to \infty$). As a result, on invoking (1.21), (2.1), (2.25) and (3.5), we find that

$$(3.8) \qquad \tilde{K}_{N,2}^{+}(n, x) = (-1)^n \frac{|2x+1|^{(2x+1)/2}\, (2n+1)^{(2n+1)/2}}{\{2e(n-x)\}^{(n+x+1)/2}} \times \left[1 + \sum_{j=1}^{N-1} (-1)^j \frac{\tilde{a}_j^{+}(\beta)}{(n-x)^j} \right]^{-1},$$

where

$$(3.9) \qquad \tilde{a}_j^{+}(\beta) = \lim_{\substack{\xi_a \to \infty \\ (s \to \infty)}} \tilde{A}_j(\xi_a).$$

The asymptotic expansion (3.7) is uniformly valid for $-\infty < x < -n-1$ and $-\{1-\delta\}\, s_0(2-\beta)(n-x) \leqslant a < \infty$.

We can compute the coefficients (3.9) by comparing both sides of (3.7) as $a \to 0$. To do this we write $U = n - x$, then from (1.29) we have $x = -\frac{1}{2}\{\beta U + 1\}$ and $n = U\{1 - \frac{1}{2}\beta\} - \frac{1}{2}$. Then we find from letting $a \to 0$ in (3.7), and referring to (1.20), (2.1), (2.24), and (3.8) that (at least formally)

$$(3.10) \qquad \frac{\Gamma(\frac{1}{2}\{\beta U + 1\})\, U^{(2U - \beta U - 1)/2}}{\Gamma(U)\, e^U} \left(\frac{2e}{\beta} \right)^{\beta U/2} \sim 1 + \sum_{j=1}^{\infty} (-1)^j \frac{\tilde{a}_j^{+}(\beta)}{U^j},$$

as $U \to \infty$. Next, by employing Stirling's formula for the Gamma functions, the left hand side of (3.10) can be expanded in inverse powers of U. This allows the calculation of the coefficients $\tilde{a}_j^{+}(\beta)$. For instance, the first three are found to be

$$(3.11) \qquad \tilde{a}_1^{+}(\beta) = \frac{1+\beta}{12\beta},$$

$$(3.12) \qquad \tilde{a}_2^{+}(\beta) = \frac{(1+\beta)^2}{288\beta^2},$$

and

$$\tilde{a}_3^+(\beta)=\frac{15\beta+15\beta^2-139\beta^3-1003}{51840\beta^3}. \tag{3.13}$$

Subcase Ib. Next, we identify solutions which are recessive at $a=0$. Thus, there exists a function $\tilde{K}(n, x)$ which is independent of a, such that

(3.14)

$$\mathbf{N}(-n, x-n+1, a)=e^{(n-x)\pi i}\tilde{K}(n, x)\, e^{a/2}(-a)^{(n-x)/2} \times\{a^2-2(n+x+1)\,a+(n-x)^2\}^{-1/4}\,\tilde{W}_{N,1}(n, x, \xi_b).$$

By letting $a\to 0-$ with $\arg(a)=\pi$ (equivalently $\xi_b\to -\infty$) and invoking (1.17), (2.1), (2.24), (3.1) and (3.3) we find that

$$\tilde{K}(n, x)=\left(\frac{2}{e}\right)^{(n-x)/2}\frac{(n-x)^{(3n-x+2)/2}}{\Gamma(n-x+1)(2n+1)^{(2n+1)/2}}. \tag{3.15}$$

The expansion (3.14) is uniformly valid for $-n-1\leqslant x<-\frac{1}{2}$ and (at least) $-\infty<s\leqslant 1-\delta$ (i.e., $-\infty<a\leqslant\{1-\delta\}(n-x)$). (If $s^+(\beta)$ is bounded away from the real axis then (3.14) is instead uniformly valid for $-\infty<a<\infty$). Since ξ_b is complex for $0<a\leqslant\{1-\delta\}(n-x)$ ($\arg(a)=0$) it is more convenient to work with real variables in this case. Thus, as an alternative expansion when a is positive, we similarly find that

(3.16)

$$\mathbf{N}(-n, x-n+1, a)=\tilde{K}(n, x)\, e^{a/2}a^{(n-x)/2} \times\{a^2-2(n+x+1)\,a+(n-x)^2\}^{-1/4}\,\tilde{W}_{N,1}(n, x, \xi_a).$$

The expansion (3.16) can be used for $-n-1\leqslant x<-\frac{1}{2}$ and $0<a\leqslant\{1-\delta\}(n-x)$.

Also for Case Ib, we match solutions which are recessive at $a=-\infty$ ($\arg(a)=\pi$). Thus we arrive at

(3.17)

$$V(-n, x-n+1, a)=\tilde{K}_{\bar{N},2}(n, x)\, e^{a/2}(-a)^{(n-x)/2} \times\{a^2-2(n+x+1)\,a+(n-x)^2\}^{-1/4}\,\tilde{W}_{\bar{N},2}(n, x, \xi_b),$$

where

$$\tilde{K}^{-}_{N,2}(n,x)=\left(\frac{2e}{n-x}\right)^{(n+x+1)/2}\left[1+\sum_{j=1}^{N-1}(-1)^j\frac{\tilde{a}^{-}_j(\beta)}{(n-x)^j}\right]^{-1}, \tag{3.18}$$

in which

$$\tilde{a}^{-}_j(\beta)=\lim_{\substack{\xi_b\to\infty\\(s\to-\infty)}}\tilde{A}_j(\xi_b). \tag{3.19}$$

The function $\tilde{K}^{-}_{N,2}(n,x)$ was found by letting $a\to-\infty$ (equivalently $\xi_b\to\infty$) in (3.17) and using (1.24), (2.1), (2.26) and (3.5). Expansion (3.17) is uniformly valid for $-n-1\leqslant x<-\frac{1}{2}$ and $-\infty<a\leqslant\{1-\delta\}\,s_0(\beta)(n-x)$.

Similar to Case Ia, we can compute the coefficients (3.19) by comparing both sides of (3.17) at $a=0$, using (1.25). Thus, from letting $a\to 0-$ in (3.17) and following the steps that lead to (3.10), we arrive at

(3.20)
$$\frac{\Gamma(\{1-\frac{1}{2}\beta\}\,U+\frac{1}{2})}{\Gamma(U)\,U^{1/2}}\left(\frac{2}{2-\beta}\right)^{U}\left(\frac{\{2-\beta\}\,U}{2e}\right)^{\beta U/2}\sim 1+\sum_{j=1}^{\infty}(-1)^j\frac{\tilde{a}^{-}_j(\beta)}{U^j}.$$

We observe that if we replace β by $2-\beta$ in the left hand side of (3.20), we get the same expression as the left hand side of (3.10). From this we deduce that, for each j,

$$\tilde{a}^{-}_j(\beta)=\tilde{a}^{+}_j(2-\beta). \tag{3.21}$$

For example, from (3.11) we find that $\tilde{a}^{-}_1(\beta)=\frac{1}{12}(3-\beta)/(2-\beta)$.

Regardless of whether x is or is not an integer, asymptotic expansions for the Charlier polynomials in Case Ib come from the above expansions for $\mathbf{N}(-n,x-n+1,a)$ and $V(-n,x-n+1,a)$ and the connection formula (1.26).

Subcase Ic. This is very similar to Case Ib. We find that the expansions (3.14), (3.16) and (3.17) still hold, but this time are uniformly valid for $-\frac{1}{2}\leqslant x\leqslant\delta^4 u-\frac{1}{2}$ and $-\infty<a\leqslant\{1-\delta\}\,s^{-}(\beta)(n-x)$.

4. CASE II: PRELIMINARY TRANSFORMATIONS

We now consider the case where $-1+\delta^4\leqslant\alpha(x)\leqslant\Delta$ (where $\alpha(x)=(x-n)/u$ and $u=n+\frac{1}{2}$), or equivalently $\delta^4 u-\frac{1}{2}\leqslant x\leqslant(1+\Delta)\,n+\frac{1}{2}\Delta$. We shall take u as the large parameter, and introduce a new independent variable t by

$$a=ut. \tag{4.1}$$

Then the differential equation (1.14) can be recast in the form

$$\frac{d^2w}{dt^2} = \{u^2 f(\alpha, t) + g(t)\}\, w, \tag{4.2}$$

where

$$g(t) = -\frac{1}{4t^2}, \tag{4.3}$$

and

$$f(\alpha, t) = \frac{t^2 - 4t - 2\alpha t + \alpha^2}{4t^2} = \frac{(t^-(\alpha) - t)(t^+(\alpha) - t)}{4t^2}, \tag{4.4}$$

in which

$$t^{\pm}(\alpha) = 2 + \alpha \pm 2\sqrt{1+\alpha}\,. \tag{4.5}$$

Thus for large u the new differential equation (4.2) has turning points at $t = t^{\pm}(\alpha)$. These turning points coalesce at $t = 1$ when $\alpha(x) \to -1$ (i.e., for $x = o(u)$), but this situation does not occur in the present case. The other critical situation, which is covered in the present case, occurs when $\alpha(x) \to 0$ (i.e., $x \to n$), since then the turning point $t^-(\alpha)$ coalesces with the double pole at $t = 0$. However, $t^-(\alpha)$ and $t^+(\alpha)$ are bounded away from one another, since for $-1 + \delta^4 \leqslant \alpha(x) \leqslant \Delta$ we have from (4.5)

$$t^+(\alpha) - t^-(\alpha) = 4\sqrt{1+\alpha} \geqslant 4\delta^2. \tag{4.6}$$

We also observe that

$$0 \leqslant t^-(\alpha) \leqslant 2 + \Delta - 2\sqrt{1+\Delta} < \infty, \tag{4.7}$$

and

$$1 < t^+(\alpha) \leqslant 2 + \Delta + 2\sqrt{1+\Delta} < \infty, \tag{4.8}$$

and hence both turning points are bounded. Our results will be uniformly valid for $-\infty < t \leqslant t^+(\alpha) - \delta$. From (4.8) we see that a sufficient condition for this to be true is $-\infty < t \leqslant 1 - \delta$ (i.e., $-\infty < a \leqslant \{1 - \delta\}\, u$).

Note that t is bounded away from $t^+(\alpha)$, but can coincide with $t^-(\alpha)$, or $t = 0$, or both simultaneously. In order to obtain asymptotic approximations which are valid for a coalescing turning point and double pole, we apply the theory of [1]. Thus, from [1, Eq. (2.1)] the appropriate Liouville transformation on (4.2) is given by

$$W = \left(\frac{d\zeta}{dt}\right)^{1/2} w, \tag{4.9}$$

and

$$\left(\frac{d\zeta}{dt}\right)^{2} = f(\alpha, t)\left(\frac{\alpha^2(x)}{4\zeta^2} - \frac{1}{4\zeta}\right)^{-1}. \tag{4.10}$$

This yields the new equation

$$\frac{d^2W}{d\zeta^2} = \left\{u^2\left(\frac{\alpha^2(x)}{4\zeta^2} - \frac{1}{4\zeta}\right) - \frac{1}{4\zeta^2} + \frac{\psi(\alpha, \zeta)}{\zeta}\right\} W, \tag{4.11}$$

where

(4.12)

$$\psi(\alpha, \zeta) = \frac{\zeta + 4\alpha^2}{16(\zeta - \alpha^2)^2} + \frac{(\zeta - \alpha^2)(5f'(\alpha, t) - 4f'(\alpha, t)\, f''(\alpha, t) - 16 f(\alpha, t)\, g(t))}{64\zeta f^3(\alpha, t)}.$$

From (4.3) and (4.4) an explicit expression for $\psi(\alpha, \zeta)$ is given by

$$\psi(\alpha, \zeta) = \frac{\zeta + 4\alpha^2}{16(\zeta - \alpha^2)^2} + \frac{(\zeta - \alpha^2)\, t(t^3 + (2 + 3\alpha)(2 - \alpha)\, t + 2\alpha^2(2 + \alpha))}{4\zeta(t^2 - 4t - 2\alpha t + \alpha^2)^3}. \tag{4.13}$$

If we map $t = t^-(\alpha)$ to $\zeta = \alpha^2$, and $t = 0$ to $\zeta = 0$, $\psi(\alpha, \zeta)$ is analytic at $\zeta = 0$ and at $\zeta = \alpha^2$.

When ζ is complex, or real with $\zeta < \alpha^2$ (equivalently $t < t^-(\alpha)$), we integrate (4.10) to give

$$\int_\zeta^{\alpha^2} \frac{(\alpha^2 - \tau)^{1/2}}{\tau}\, d\tau = \int_t^{t^-(\alpha)} \frac{[(t^-(\alpha) - q)(t^+(\alpha) - q)]^{1/2}}{q}\, dq. \tag{4.14}$$

The integrals can be explicitly evaluated to give the relation

(4.15)

$$\alpha \ln\left\{\frac{2\alpha^2 - \zeta + 2\alpha\sqrt{\alpha^2 - \zeta}}{\zeta}\right\} - 2\sqrt{\alpha^2 - \zeta}$$
$$= (2 + \alpha)\ln\left\{\frac{2 + \alpha - t - \sqrt{(t^-(\alpha) - t)(t^+(\alpha) - t)}}{2\sqrt{1 + \alpha}}\right\} - \sqrt{(t^-(\alpha) - t)(t^+(\alpha) - t)}$$
$$+ \alpha \ln\left\{\frac{\alpha^2 - \alpha t - 2t + \alpha\sqrt{(t^-(\alpha) - t)(t^+(\alpha) - t)}}{2t\sqrt{1 + \alpha}}\right\}.$$

The branches are defined such that ζ is real and lying in $(-\infty, \alpha^2)$ when t is real and lying in $(-\infty, t^-(\alpha))$, and by continuity for complex values of the variables.

When ζ is real with $\zeta \geqslant \alpha^2$ (equivalently $t \geqslant t^-(\alpha)$) the map is given by

$$\int_{\alpha^2}^{\zeta} \frac{(\tau-\alpha^2)^{1/2}}{\tau}\, d\tau = \int_{t^-(\alpha)}^{t} \frac{[(q-t^-(\alpha))(t^+(\alpha)-q)]^{1/2}}{q}\, dq, \tag{4.16}$$

which on integration yields

$$\begin{aligned} 2\sqrt{\zeta-\alpha^2} - 2\alpha \arccos\left\{\frac{\alpha}{\sqrt{\zeta}}\right\} &= \sqrt{(t-t^-(\alpha))(t^+(\alpha)-t)} \\ &\quad + \{2+\alpha\} \arccos\left\{\frac{2+\alpha-t}{\sqrt{1+\alpha}}\right\} \\ &\quad - \alpha \arccos\left\{\frac{\alpha^2-\alpha t-2t}{2t\sqrt{1+\alpha}}\right\}. \end{aligned} \tag{4.17}$$

Here the inverse cosines are nonnegative, are equal to 0 for $\zeta = \alpha^2$ $(t = t^-(\alpha))$, and are defined by continuity for other values of ζ and t. The value of ζ corresponding to $t = t^+(\alpha)$ is $\zeta = \zeta^+(\alpha)$ say, where $\zeta^+(\alpha) > \alpha^2$. From (4.17) we see that for each $\alpha \in [-1, \infty)$ it is implicitly defined as the unique positive solution of

$$\sqrt{\zeta^+(\alpha)-\alpha^2} - \alpha \arccos\left\{\frac{\alpha}{\sqrt{\zeta^+(\alpha)}}\right\} = \pi, \tag{4.18}$$

where again the inverse cosine is nonnegative. So, for example, $\zeta^+(-1) = 1$, $\zeta^+(0) = \pi^2$, and $\zeta^+(1) = 21.1907\ldots$. We note that $\zeta^+(\alpha)$ is an increasing function of $\alpha \in [-1, \infty)$, since implicit differentiation of (4.18) gives

$$\frac{d\zeta^+(\alpha)}{d\alpha} = \frac{2\zeta^+(\alpha)}{\sqrt{\zeta^+(\alpha)-\alpha^2}} \arccos\left\{\frac{\alpha}{\sqrt{\zeta^+(\alpha)}}\right\} > 0. \tag{4.19}$$

Next, from (4.15), we find the behaviour

$$t = \frac{e}{4(1+\alpha)^{1+1/\alpha}}\zeta + O(\zeta^2) \tag{4.20}$$

as $\zeta \to 0$. We shall use this in our identification of solutions of (4.11). It is also straightforward to show from (4.15) that

(4.21)
$$t = -2\sqrt{-\zeta} + \tfrac{1}{2}(2+\alpha)\ln(-4\zeta) + 2 + \alpha - (1+\alpha)\ln(1+\alpha) + O(\ln(\zeta)/\sqrt{\zeta})$$

as $\zeta \to -\infty$. From this and (4.13) we observe that

$$\psi(\alpha, \zeta) = \frac{(2+\alpha) \ln |\zeta|}{32 |\zeta|^{3/2}} + O(|\zeta|^{-3/2}), \tag{4.22}$$

as $\zeta \to -\infty$.

5. CASE II: ASYMPTOTIC EXPANSIONS

From [1, Theorem 1] we obtain an asymptotic solution for $\zeta > 0$ to (4.11) of the form

$$\begin{aligned} W_{2N+1,1}(u, \alpha, \zeta) = {} & \zeta^{1/2} J_{u|\alpha|}(u\zeta^{1/2}) \sum_{j=0}^{N} \frac{A_j(\alpha, \zeta)}{u^{2j}} \\ & + \frac{\zeta}{u} J'_{u|\alpha|}(u\zeta^{1/2}) \sum_{j=0}^{N-1} \frac{B_j(\alpha, \zeta)}{u^{2j}} \\ & + \varepsilon_{2N+1,1}(u, \alpha, \zeta), \end{aligned} \tag{5.1}$$

which is recessive at $\zeta = 0$ ($t = 0$). The coefficients are given recursively by $A_0(\alpha, \zeta) = 1$, and

(5.2)

$$B_j(\alpha, \zeta) = |\zeta - \alpha^2|^{-1/2} \int_{\alpha^2}^{\zeta} |\tau - \alpha^2|^{-1/2} \{\tau A_j''(\alpha, \tau) + A_j'(\alpha, \tau) - \psi(\alpha, \tau) A_j(\alpha, \tau)\} \, d\tau$$

and

$$A_j(\alpha, \zeta) = -\zeta B_{j-1}'(\alpha, \zeta) + \int \psi(\alpha, \zeta) B_{j-1}(\alpha, \zeta) \, d\zeta. \tag{5.3}$$

The integration constants associated with (5.3) can be arbitrarily chosen: see the paragraph after Eq. (5.25) below. A bound on the error term $\varepsilon_{2N+1,1}(u, \alpha, \zeta)$ is given by [1, Eq. (3.8)], and for our case is uniformly valid for $0 < \zeta \leqslant \zeta^{+}(\alpha) - \delta$.

We note the behaviour

(5.4)

$$W_{2N+1,1}(u, \alpha, \zeta) \sim \frac{\zeta^{(u|\alpha|+1)/2}}{\Gamma(u|\alpha|+1)} \left(\frac{u}{2}\right)^{u|\alpha|} \left[1 + \sum_{j=1}^{N} \frac{A_j(\alpha, 0)}{u^{2j}} + |\alpha| \sum_{j=0}^{N-1} \frac{B_j(\alpha, 0)}{u^{2j+1}}\right],$$

as $\zeta \to 0$.

For $\zeta<0$ the real-valued asymptotic solution which is recessive at $\zeta=0$ is given by

$$(5.5)\qquad W_{2N+1,3}(u,\alpha,\zeta)=|\zeta|^{1/2}\,I_{u|\alpha|}(u\,|\zeta|^{1/2})\sum_{j=0}^{N}\frac{A_j(\alpha,\zeta)}{u^{2j}}+\frac{|\zeta|}{u}\,I'_{u|\alpha|}(u\,|\zeta|^{1/2})\sum_{j=0}^{N-1}\frac{B_j(\alpha,\zeta)}{u^{2j}}+\varepsilon_{2N+1,3}(u,\alpha,\zeta).$$

Also for $\zeta<0$, the real-valued solution which is recessive at $\zeta=-\infty$ is given by

$$(5.6)\qquad W_{2N+1,4}(u,\alpha,\zeta)=|\zeta|^{1/2}\,K_{u|\alpha|}(u\,|\zeta|^{1/2})\sum_{j=0}^{N}\frac{A_j(\alpha,\zeta)}{u^{2j}}+\frac{|\zeta|}{u}\,K'_{u|\alpha|}(u\,|\zeta|^{1/2})\sum_{j=0}^{N-1}\frac{B_j(\alpha,\zeta)}{u^{2j}}+\varepsilon_{2N+1,4}(u,\alpha,\zeta).$$

Bounds on the error terms $\varepsilon_{2N+1,3}(u,\alpha,\zeta)$ and $\varepsilon_{2N+1,4}(u,\alpha,\zeta)$ are given by [1, Eqs. (3.14) and (3.15)], and are uniformly valid for $-\infty<\zeta<0$.

It can be shown that as $\zeta\to 0-$ (with $\alpha\neq 0$)

(5.7)

$$W_{2N+1,4}(u,\alpha,\zeta)\sim\frac{1}{2}\,|\zeta|^{(1-u|\alpha|)/2}\left(\frac{2}{u}\right)^{u|\alpha|}\Gamma(u\,|\alpha|)\times\left[1+\sum_{j=1}^{N}\frac{A_j(\alpha,0)}{u^{2j}}-|\alpha|\sum_{j=0}^{N-1}\frac{B_j(\alpha,0)}{u^{2j+1}}+\delta_{2N+1,4}(u,\alpha)\right],$$

where

$$(5.8)\qquad \delta_{2N+1,4}(u,\alpha)=\lim_{\zeta\to 0-}\{|\zeta|^{1/2}\,K_{u|\alpha|}(u\,|\zeta|^{1/2})\}^{-1}\,\varepsilon_{2N+1,4}(u,\alpha,\zeta).$$

To obtain an expansion for $V(-n,x-n+1,a)$ for the case $\zeta>0$ $(a>0)$ we must resort to a complex-valued asymptotic solution which is recessive at $\zeta=-\infty$ $(\arg(\zeta)=\pi)$. This is given by [1, Theorem 3]:

$$(5.9)\qquad W^{(1)}_{2N+1}(u,\alpha,\zeta)=\zeta^{1/2}H^{(1)}_{u|\alpha|}(u\zeta^{1/2})\sum_{j=0}^{N}\frac{A_j(\alpha,\zeta)}{u^{2j}}+\frac{\zeta}{u}\,H^{(1)\prime}_{u|\alpha|}(u\zeta^{1/2})\sum_{j=0}^{N-1}\frac{B_j(\alpha,\zeta)}{u^{2j}}+\varepsilon^{(1)}_{2N+1}(u,\alpha,\zeta).$$

Here ζ is regarded as complex, and hence in place of (5.2) we must use

$$B_j(\alpha, \zeta) = (\zeta-\alpha^2)^{-1/2} \int_{\alpha^2}^{\zeta} (\tau-\alpha^2)^{-1/2} \times \{\tau A_j''(\alpha, \tau)+A_j'(\alpha, \tau)-\psi(\alpha, \tau)\, A_j(\alpha, \tau)\}\, d\tau, \tag{5.10}$$

which are real for ζ real, and defined by continuity for complex ζ. The $A_j(\alpha, \zeta)$ are again given by (5.3) when ζ is complex. The error term $\varepsilon_{2N+1}^{(1)}(u, \alpha, \zeta)$ is bounded by [1, Eq. (5.16)]. In the present case, the bound is uniformly valid for $0<\zeta \leqslant \zeta^{+}(\alpha)-\delta$ ($\arg(\zeta)=0$), and for $-\infty<\zeta<0$ ($\arg(\zeta)=\pi$).

We will use later the fact that as $\zeta \to 0+$

$$W_{2N+1}^{(1)}(u, \alpha, \zeta) \sim -\frac{i\Gamma(u\,|\alpha|)}{\pi}\left(\frac{2}{u}\right)^{u\,|\alpha|} \zeta^{(1-u\,|\alpha|)/2} \times\left[1+\sum_{j=1}^{N} \frac{A_j(\alpha, 0)}{u^{2j}}-|\alpha| \sum_{j=0}^{N-1} \frac{B_j(\alpha, 0)}{u^{2j+1}}+\delta_{2N+1}^{(1)}(u, \alpha)\right], \tag{5.11}$$

where

$$\delta_{2N+1}^{(1)}(u, \alpha) = \lim_{\zeta \to 0} \{\zeta^{1/2} H_{u\,|\alpha|}^{(1)}(u\zeta^{1/2})\}^{-1}\, \varepsilon_{2N+1}^{(1)}(u, \alpha, \zeta). \tag{5.12}$$

Subcase IIa. Here $\delta^4 u-\frac{1}{2} \leqslant x<n$. In this case we note that $|\alpha(x)| = -\alpha(x)=(n-x)/u$. As in Cases Ib and Ic, we will obtain asymptotic expansions for $\mathbf{N}(-n, x-n+1, a)$ (recessive at $a=0$) and $V(-n, x-n+1, a)$ (recessive at $a=-\infty$), and then appeal to the connection formula (1.26) to obtain the corresponding (compound) asymptotic expansions for the Charlier polynomials.

First, matching solutions which are recessive at $\zeta=0$ ($a=0$), we find for $\zeta>0$ ($a>0$) that

$$\mathbf{N}(-n, x-n+1, a) = \mathbf{K}_{2N+1,\,1}(n, x)\, e^{a/2} a^{(n-x)/2} \times\left(\frac{\alpha^2-\zeta}{(x-n)^2+a^2-2(x+n-1)\, a}\right)^{1/4} \zeta^{-1/2} W_{2N+1,\,1}(u, \alpha, \zeta), \tag{5.13}$$

where (by using (1.17), (4.1), (4.20) and (5.4))

$$\mathbf{K}_{2N+1,\,1}(n, x) = \sqrt{u}\,(1+\alpha)^{x/2+1/4}\left(\frac{e}{u}\right)^{(n-x)/2} \times\left[1+\sum_{j=1}^{N} \frac{A_j(\alpha, 0)}{u^{2j}}+|\alpha| \sum_{j=0}^{N-1} \frac{B_j(\alpha, 0)}{u^{2j+1}}\right]^{-1}. \tag{5.14}$$

This expansion is uniformly valid for $\delta^4 u - \frac{1}{2} \leqslant x < n$ and $0 < \zeta \leqslant \zeta^+(\alpha) - \delta$ (i.e., $0 < a \leqslant \{t^+(\alpha) - \delta\}\, u$), where δ is an arbitrary positive constant.

For $\zeta < 0$ ($a < 0$ with $\arg(a) = \pi$) we similarly find that

$$(5.15) \quad \mathbf{N}(-n, x-n+1, a) = e^{(n-x)\pi i} \mathbf{K}_{2N+1,3}(n, x)\, e^{a/2} |\alpha|^{(n-x)/2} \times \left(\frac{\alpha^2 - \zeta}{(x-n)^2 + a^2 - 2(x+n+1)\, a} \right)^{1/4} |\zeta|^{-1/2} W_{2N+1,3}(u, \alpha, \zeta),$$

where $\mathbf{K}_{2N+1,3}(n, x) = \mathbf{K}_{2N+1,1}(n, x)$. Also for $\zeta < 0$ ($a < 0$ with $\arg(a) = \pi$), we find by matching solutions which are recessive at $\zeta = -\infty$ ($a = -\infty$) that

$$(5.16) \quad V(-n, x-n+1, a) = K_{2N+1,4}(n, x)\, e^{a/2} |a|^{(n-x)/2} \times \left(\frac{\alpha^2 - \zeta}{(x-n)^2 + a^2 - 2(x+n+1)\, a} \right)^{1/4} |\zeta|^{-1/2} W_{2N+1,4}(u, \alpha, \zeta).$$

Now comparing both sides of (5.16) as $a \to 0-$, we find from (1.25), (4.1), (4.20) and (5.7) that

$$(5.17) \quad K_{2N+1,4}(n, x) = \frac{2\sqrt{u}}{n!(1+\alpha)^{x/2+1/4}} \left(\frac{u}{e} \right)^{(n-x)/2} \times \left[1 + \sum_{j=1}^{N} \frac{A_j(\alpha, 0)}{u^{2j}} - |\alpha| \sum_{j=0}^{N-1} \frac{B_j(\alpha, 0)}{u^{2j+1}} + \delta_{2N+1,4}(u, \alpha) \right]^{-1},$$

in which $\delta_{2N+1,4}(u, \alpha)$ is given by (5.8). The expansions (5.15) and (5.16) are both uniformly valid for $\delta^4 u - \frac{1}{2} \leqslant n$ and $-\infty < \zeta < 0$ ($-\infty < a < 0$).

In order to obtain an expansion for $V(-n, x-n+1, a)$ which is valid for $\zeta > 0$ ($a > 0$), we match it with the complex-valued asymptotic solution (5.9) which is recessive at $\zeta = -\infty$ ($\arg(\zeta) = \pi$). Thus we can assert that

$$(5.18) \quad V(-n, x-n+1, a) = iK^{(1)}_{2N+1}(n, x)\, e^{a/2} a^{(n-x)/2} \times \left(\frac{\alpha^2 - \zeta}{(x-n)^2 + a^2 - 2(x+n+1)\, a} \right)^{1/4} \zeta^{-1/2} W^{(1)}_{2N+1}(u, \alpha, \zeta),$$

for some function $K^{(1)}_{2N+1}(n, x)$ which is independent of a. Here ζ is defined as the complex-valued solution of (4.15). Now from (1.25), (4.1), (4.20), and (5.11) we find from letting $a \to 0+$ in (5.18) that

$$K^{(1)}_{2N+1}(n, x) = \frac{\pi\sqrt{u}}{n!(1+\alpha)^{x/2+1/4}}\left(\frac{u}{e}\right)^{(n-x)/2} \times\left[1+\sum_{j=1}^{N}\frac{A_j(\alpha, 0)}{u^{2j}} - |\alpha|\sum_{j=0}^{N-1}\frac{B_j(\alpha, 0)}{u^{2j+1}} + \delta^{(1)}_{2N+1}(u, \alpha)\right]^{-1}, \tag{5.19}$$

where $\delta^{(1)}_{2N+1}(u, \alpha)$ is given by (5.12). If we now restrict a and ζ to being positive (with $\arg(a) = \arg(\zeta) = 0$), then (5.18) is uniformly valid for $\delta^4 u - \frac{1}{2} \leqslant x < n$ and $0 < \zeta \leqslant \zeta^+(\alpha) - \delta$ (i.e., $0 < a \leqslant \{t^+(\alpha) - \delta\}\, u$).

Asymptotic expansions for the Charlier polynomials now come from the above expansions for $\mathbf{N}(-n, x-n+1, a)$ and $V(-n, x-n+1, a)$ and the connection formula (1.26). For $a < 0$ we use (5.15) and (5.16), and for $a > 0$ we use (5.13) and (5.18). We note that when a (equivalently t and ζ) is positive, imaginary terms appear in the compound expansion for $C^{(a)}_n(x)$. Of course $C^{(a)}_n(x)$ is real, and so we can drop the imaginary components in the compound expansion (which necessarily must vanish identically). Therefore, using (1.26), (5.15) and (5.16) we arrive at the compound asymptotic expansion

(5.20)

$$\begin{aligned} C^{(a)}_n(x) &= n!\, e^{a/2} a^{(n-x)/2}\left(\frac{\alpha^2 - \zeta}{(x-n)^2 + a^2 - 2(x+n+1)\, a}\right)^{1/4} \zeta^{-1/2} \\ &\times\Bigg[\cos\{(n-x)\,\pi\}\,\mathbf{K}_{2N+1,1}(n, x)\, W_{2N+1,3}(u, \alpha, \zeta) \\ &-\frac{(-1)^n}{\Gamma(-x)} K^{(1)}_{2N+1}(n, x) \\ &\times\Bigg\{\zeta^{1/2} Y_{n-x}(u\zeta^{1/2})\sum_{j=0}^{N}\frac{A_j(\alpha, \zeta)}{u^{2j}} + \frac{\zeta}{u} Y'_{n-x}(u\zeta^{1/2})\sum_{j=0}^{N-1}\frac{B_j(\alpha, \zeta)}{u^{2j}} \\ &+ \operatorname{Im}\varepsilon^{(1)}_{2N+1}(u, \alpha, \zeta)\Bigg\}\Bigg], \end{aligned}$$

which is uniformly valid for $\delta^4 u - \frac{1}{2} \leqslant x < n$ and $0 < \zeta \leqslant \zeta^+(\alpha) - \delta$ (i.e., $0 < a \leqslant \{t^+(\alpha) - \delta\}\, u$).

Let us analyse the vanishing of the imaginary component further. This will give us a means of evaluating $A_s(\alpha, 0)$ and $B_s(\alpha, 0)$, which appear in the formulas for $\mathbf{K}_{2N+1,1}(n, x)$ and $K^{(1)}_{2N+1}(n, x)$. The vanishing of the imaginary component in the compound expansion (5.20) occurs either for x an integer, or for

(5.21)

$$\left\{\frac{\Gamma(x+1)}{\pi} K^{(1)}_{2N+1}(n, x) - \mathbf{K}_{2N+1,1}(n, x)\right\}$$
$$\times\left\{\zeta^{1/2} J_{n-x}(u\zeta^{1/2}) \sum_{j=0}^{N} \frac{A_j(\alpha, \zeta)}{u^{2j}} + \frac{\zeta}{u} J'_{n-x}(u\zeta^{1/2}) \sum_{j=0}^{N-1} \frac{B_j(\alpha, \zeta)}{u^{2j}}\right\}$$
$$= \mathbf{K}_{2N+1,1}(n, x)\, \varepsilon_{2N+1,1}(u, \alpha, \zeta) - \frac{\Gamma(x+1)}{\pi} K^{(1)}_{2N+1}(n, x) \operatorname{Re} \varepsilon^{(1)}_{2N+1}(u, \alpha, \zeta),$$

when x is not an integer. Therefore, if we temporarily assume that x is not an integer, we deduce that for each N

$$\mathbf{K}_{2N+1,1}(n, x) = \frac{\Gamma(x+1)}{\pi} K^{(1)}_{2N+1}(n, x)\left\{1 + O\left(\frac{1}{u^{2N+1}}\right)\right\}, \tag{5.22}$$

since (5.21) must hold for all $\zeta \in (0, \zeta^+(\alpha) - \delta]$. Hence, since N is arbitrary, we deduce from (5.14), (5.19) and (5.22) the formal relationship

(5.23)

$$(1+\alpha)^{x/2+1/4}\left(\frac{e}{u}\right)^{(n-x)/2}\left[1 + \sum_{j=1}^{\infty} \frac{A_j(\alpha, 0)}{u^{2j}} + |\alpha| \sum_{j=0}^{\infty} \frac{B_j(\alpha, 0)}{u^{2j+1}}\right]^{-1}$$
$$\sim \frac{\Gamma(x+1)}{n!(1+\alpha)^{x/2+1/4}}\left(\frac{u}{e}\right)^{(n-x)/2}\left[1 + \sum_{j=1}^{\infty} \frac{A_j(\alpha, 0)}{u^{2j}} - |\alpha| \sum_{j=0}^{\infty} \frac{B_j(\alpha, 0)}{u^{2j+1}}\right]^{-1}.$$

Next, we write $x = u + \alpha u - \frac{1}{2}$ and $n = u - \frac{1}{2}$, and arrive at the expansion

(5.24)

$$\frac{\Gamma(u+\frac{1}{2})}{\Gamma(u+u\alpha+\frac{1}{2})}\left(\frac{u+u\alpha}{e}\right)^{u\alpha}(1+\alpha)^u$$
$$\sim\left[1 + \sum_{j=1}^{\infty} \frac{A_j(\alpha, 0)}{u^{2j}} - \alpha \sum_{j=0}^{\infty} \frac{B_j(\alpha, 0)}{u^{j+1}}\right]\left[1 + \sum_{j=1}^{\infty} \frac{A_j(\alpha, 0)}{u^{2j}} + \alpha \sum_{j=0}^{\infty} \frac{B_j(\alpha, 0)}{u^{2j+1}}\right]^{-1},$$

(recalling that $|\alpha| = -\alpha$ in this case). On appealing to Stirling's formula, the left hand side has the asymptotic expansion (for α not close to -1, as is the case here)

(5.25)

$$\frac{\Gamma(u+\frac{1}{2})}{\Gamma(u+u\alpha+\frac{1}{2})}\left(\frac{u+u\alpha}{e}\right)^{u\alpha}(1+\alpha)^u$$
$$\sim 1 - \frac{\alpha}{24(1+\alpha)\,u} + \frac{\alpha^2}{1152(1+\alpha)^2\,u^2} + \frac{\alpha(1003\alpha^2 + 3024\alpha + 3024)}{414720(1+\alpha)^3\,u^3} + \cdots.$$

The restriction x being a non-integer can now be relaxed. Comparison of the right hand sides of the expansions (5.24) and (5.25) allows us to compute the $B_j(\alpha, 0)$ in terms of the $A_j(\alpha, 0)$, which themselves can be arbitrarily chosen due to the integration constants in (5.3). For example, a natural choice would be $A_j(\alpha, 0)=0$ for $j=1, 2, 3, \ldots$, and then from comparing (5.24) and (5.25) we find that

$$B_0(\alpha, 0)=\frac{48}{(1+\alpha)}, \tag{5.26}$$

$$B_1(\alpha, 0)=-\frac{6048+6048\alpha+2021\alpha^2}{1658880(1+\alpha)^3}, \tag{5.27}$$

$$B_2(\alpha, 0)=\frac{7\alpha^2(3+3\alpha+\alpha^2)}{4423680(1+\alpha)^5}, \tag{5.28}$$

and so on.

Subcase IIb. Here $n \leqslant x \leqslant (1+\Delta)\, n+\frac{1}{2}\Delta$ (and hence $|\alpha(x)|=\alpha(x)$). In this case $C_n^{(a)}(x)$ is recessive at $a=0$, and consequently for $\zeta>0$ we make the following direct identification with the corresponding asymptotic solution (5.1)

$$C_n^{(a)}(x)=K_{2N+1,1}(n, x)\, e^{a/2} a^{-(x-n)/2} \times\left(\frac{\alpha^2-\zeta}{(x-n)+a^2-2(x+n+1)\, a}\right)^{1/4} \zeta^{-1/2} W_{2N+1,1}(u, \alpha, \zeta). \tag{5.29}$$

Letting $\zeta \to 0$ and using (1.15), (4.1), (4.20), and (5.4) we find that

$$K_{2N+1,1}(n, x)=\frac{\sqrt{u}\, \Gamma(x+1)}{(1+\alpha)^{x/2+1/4}}\left(\frac{e}{u}\right)^{(x-n)/2} \times\left[1+\sum_{j=1}^{N} \frac{A_j(\alpha, 0)}{u^{2j}}+\alpha \sum_{j=0}^{N-1} \frac{B_s(\alpha, 0)}{u^{2j+1}}\right]^{-1}. \tag{5.30}$$

The expansion (5.29) is uniformly valid for $n \leqslant x \leqslant (1+\Delta)\, n+\frac{1}{2}\Delta$ and $0<a \leqslant \{t^{+}(\alpha)-\delta\}\, u$.

Similarly, for $\zeta<0$ $(a<0)$ we obtain

$$C_n^{(a)}(x)=K_{2N+1,3}(n, x)\, e^{a/2}\, |a|^{-(x-n)/2} \tag{5.31}$$

$$\times\left(\frac{\alpha^2-\zeta}{(x-n)^2+a^2-2(x+n+1)\, a}\right)^{1/4} |\zeta|^{-1/2}\, W_{2N+1,3}(u, \alpha, \zeta),$$

where it is readily shown that $K_{2N+1,3}(n, x) = K_{2N+1,1}(n, x)$. This asymptotic expansion is uniformly valid for $n \leqslant x \leqslant (1+\Delta)\, n + \frac{1}{2}\Delta$ and $-\infty < a < 0$.

6. CASE III: PRELIMINARY TRANSFORMATIONS

Here $\Delta \leqslant \alpha(x) < \infty$, which from (1.28) is seen to be equivalent to $(1+\Delta)\, n + \frac{1}{2}\Delta \leqslant x < \infty$, and so now we shall take $x+\frac{1}{2}$ as the large asymptotic parameter. We then, for convenience, introduce the parameter $v(x)$ by (1.32) above. In terms of $\alpha(x)$ we have

(6.1)
$$v(x) = \{1+\alpha(x)\}^{-1/2}.$$

The new independent variable is this time defined by

(6.2)
$$r = \frac{a}{x+\frac{1}{2}}.$$

Then, we rewrite (1.14) in the form

(6.3)
$$\frac{d^2 w}{dr^2} = \left\{ \left(x + \frac{1}{2} \right)^2 \hat{f}(r) + \hat{g}(r) \right\} w,$$

where

(6.4)
$$\hat{f}(r) = \frac{((1-v)^2 - r)((1+v)^2 - r)}{4r^2},$$

and

(6.5)
$$\hat{g}(r) = -\frac{1}{4r^2}.$$

From (6.4) we observe two turning points, at $r = (1 \pm v)^2$. Now for $\Delta \leqslant \alpha(x) < \infty$ we see from (6.1) that

(6.6)
$$0 < v(x) \leqslant \frac{1}{\sqrt{1+\Delta}}.$$

Therefore the two turning points are bounded, and can coalesce at $r = 1$ (for $v(x) \to 0$, or equivalently $n = o(x)$). However, the smaller turning point is bounded away from the pole at $r = 0$, since

(6.7)
$$(1-v)^2 \geqslant 1 + \frac{1}{1+\Delta} - \frac{2}{\sqrt{1+\Delta}} \geqslant \frac{1}{1+\Delta};$$

recall $\Delta \geqslant 3$. Our results will be uniformly valid for $-\infty < r \leqslant \{1-\delta\}(1-v)^2$. Thus, the independent variable r is bounded away from the smaller (and also, of course, the larger) turning point. The condition $-\infty < a \leqslant \{1-\delta\} u$ is sufficient for this, since from (6.7) and $x \geqslant (1+\Delta)\, u - \frac{1}{2}$, we have

$$r = \frac{a}{x+\frac{1}{2}} \leqslant \frac{1-\delta}{1+\Delta} \leqslant \{1-\delta\}(1-v)^2. \tag{6.8}$$

Since there are no turning points in the interval under consideration, we use the same general theory as in Case I, namely the Liouville–Green (WKBJ) approximation. Therefore the appropriate Liouville transformation is again given by [5, Chap. 10], and to avoid confusion with the notation of Case I, we let η denote the new independent variable of the Liouville transformation. Therefore, from (6.4) and [5, Chap. 10, Eq. (2.02)] we have

$$\eta = \int \frac{\sqrt{((1-v)^2-r)((1+v)^2-r)}}{2r}\, dr, \tag{6.9}$$

which on explicit integration yields

$$\begin{aligned} \eta = {} & \tfrac{1}{2}\sqrt{((1-v)^2-r)((1+v)^2-r)} + \tfrac{1}{2}(1-v^2)\ln|r| \\ & -\tfrac{1}{2}(1+v^2)\ln\{1+v^2-r-\sqrt{((1-v)^2-r)((1+v)^2-r)}\} \\ & -\tfrac{1}{2}(1-v^2)\ln\{(1-v^2)^2-(1+v^2)\, r \\ & +(1-v^2)\sqrt{((1-v)^2-r)((1+v)^2-r)}\} \end{aligned} \tag{6.10}$$

We consider the cases $r > 0$ and $r < 0$ separately. In the former case the r interval $(0, (1-v)^2 - \delta/(1+\Delta)]$ is mapped 1–1 to the η interval $(-\infty, \eta_0]$, where

$$\eta_0 = \eta((1-v)^2 - \delta/(1+\Delta)) = -\ln(2v) + O(\delta^{3/2}). \tag{6.11}$$

In the latter case the r interval $(-\infty, 0)$ is mapped 1–1 to the η interval $(-\infty, \infty)$. In both cases $r = 0$ corresponds to $\eta = -\infty$, and in the second case $r = -\infty$ corresponds to $\eta = \infty$.

Now $\eta \to -\infty$ as $r \to 0$, such that

$$\begin{aligned} \eta = {} & \tfrac{1}{2}(1-v^2)\ln|r| + \tfrac{1}{2}(1-v^2) - (1+v^2)\ln(v) \\ & -\ln(2) - (1-v^2)\ln(1-v^2) + O(r). \end{aligned} \tag{6.12}$$

Next, with the new dependent variable

$$\hat{W} = \hat{f}^{1/4}(r)\, w, \tag{6.13}$$

we arrive at

$$\frac{d^2\hat{W}}{d\eta^2} = \left\{\left(x+\frac{1}{2}\right)^2 + \hat{\psi}(\eta)\right\}\hat{W}, \tag{6.14}$$

where $\tilde{\psi}(\eta)$ is given by

$$\begin{aligned}\hat{\psi}(\eta) = {} & \{4r^4 - 32(1+v^2)\, r^3 + 4(15-2v^2+15v^4)\, r^2 \\ & -40(1-v^2)^2\,(1+v^2)\, r + 8(1-v^2)^4\} \\ & \times((1-v)^2-r)^{-3}\,((1+v)^2-r)^{-3}.\end{aligned} \tag{6.15}$$

From (6.12) we note that $\hat{\psi}(\eta)$ is exponentially small (in η) as $\eta \to -\infty$ $(r \to 0)$. It is also straightforward to show that $\hat{\psi}(\eta) = O(\eta^{-2})$ as $\eta \to \infty$ $(r \to -\infty)$.

7. CASE III: ASYMPTOTIC EXPANSIONS

From [5, Chap. 10, Theorem 3.1] we obtain the asymptotic solution

$$\hat{W}_{N,1}(x, \eta) = e^{(x+1/2)\eta}\left[1 + \sum_{j=1}^{N-1} \frac{\hat{A}_j(\eta)}{(x+\frac{1}{2})^j}\right] + \hat{\varepsilon}_{N,1}(x, \eta), \tag{7.1}$$

which is recessive as $\eta \to -\infty$ (i.e., $r \to 0$, or $a \to 0$). The coefficients are given recursively by

$$\hat{A}_{j+1}(\eta) = -\tfrac{1}{2}\hat{A}_j'(\eta) + \tfrac{1}{2}\int_{-\infty}^{\eta} \hat{\psi}(\tau)\,\hat{A}_j(\tau)\, d\tau \qquad (j = 0, 1, 2, \ldots), \tag{7.2}$$

with $\hat{A}_0(\eta) = 1$. The lower integration limit is chosen so that

$$\lim_{\eta \to -\infty} \hat{A}_j(\eta) = 0 \qquad (j = 1, 2, 3 \ldots). \tag{7.3}$$

The error term satisfies the bound

$$|\hat{\varepsilon}_{N,1}(x, n)| \leqslant 2e^{(x+1/2)\eta} \exp\left\{\frac{2\mathscr{V}_{-\infty,\eta}(\hat{A}_1)}{x+\frac{1}{2}}\right\} \frac{\mathscr{V}_{-\infty,\eta}(\hat{A}_N)}{(x+\frac{1}{2})^N}, \tag{7.4}$$

which holds for $\eta \in (-\infty, \eta_0]$ when $r < 0$, and for $\eta \in (-\infty, \infty)$ when $r < 0$.

We now match solutions which are recessive at $\eta=-\infty$ and $r=0$, and as such we have

$$C_n^{(a)}(x)=\hat{K}(n,x)\,e^{a/2}a^{-(1+x-n)/2}\{4\hat{f}(r)\}^{-1/4}\,\hat{W}_{N,1}(x,\eta), \tag{7.5}$$

where $\hat{K}(n,x)$ (which is independent of r and a) can conveniently be determined by

$$\hat{K}(n,x)=\lim_{\substack{\eta\to-\infty\\(r\to 0)}}\frac{e^{-a/2}a^{(1+x-n)/2}\{4\hat{f}(r)\}^{1/4}\,C_n^{(a)}(x)}{\hat{W}_{N,1}(x,\eta)}. \tag{7.6}$$

Hence from (1.15), (3.2), (6.4), (6.12), (7.1) and (7.3) we find that

$$\hat{K}(n,x)=\frac{\Gamma(x+1)(n+\frac{1}{2})^{(x+n+1)/2}\,(x-n)^{x-n+1/2}}{\Gamma(x-n+1)\,e^{(x-n)/2}}\left(\frac{2}{x+\frac{1}{2}}\right)^{x+1/2}. \tag{7.7}$$

In terms of the original variables, the expansion (7.5) is uniformly valid for $(1+\Delta)\,n+\frac{1}{2}\Delta\leqslant x<\infty$ and $0<a\leqslant\{1-\delta\}(1-v(x))^2\,(x+\frac{1}{2})$.

The case $-\infty<a\leqslant 0$ is treated similarly, and we find that (7.5) also holds uniformly for $(1+\Delta)\,n+\frac{1}{2}\Delta\leqslant x<\infty$ and $-\infty<a<0$.

8. SUMMARY OF MAIN RESULTS

Case I. This case provides asymptotic expansions for $-\infty<x\leqslant\delta^4u-\frac{1}{2}$, where $u=n+\frac{1}{2}$. Let $s=a/(n-x)$ and $\beta(x)=-(2x+1)/(n-x)$, and define Liouville variables ξ_a and ξ_b by (2.18) and (2.19). Then for each nonnegative integer N and for $\xi=\xi_a$ or $\xi=\xi_b$, the following Liouville–Green asymptotic expansion

$$\tilde{W}_{N,1}(n,x,\xi)=e^{(n-x)\xi}\left[1+\sum_{j=1}^{N-1}\frac{\tilde{A}_j(\xi)}{(n-x)^j}\right]+\tilde{\varepsilon}_{N,1}(n,x,\xi) \tag{8.1}$$

is recessive at $a=0$ ($\xi=\xi_a=\xi_b=-\infty$), and the following asymptotic expansions

$$\tilde{W}^{\pm}_{N,2}(n,x,\xi)=e^{-(n-x)\xi}\left[1+\sum_{j=1}^{N-1}(-1)^j\frac{\tilde{A}_j(\xi)}{(n-x)^j}\right]+\tilde{\varepsilon}^{\pm}_{N,2}(n,x,\xi) \tag{8.2}$$

are recessive at $a=+\infty$ ($\xi_a=+\infty$, for the plus superscript) and $a=-\infty$ ($\xi_b=+\infty$, for the minus superscript). The coefficients $\tilde{A}_j(\xi)$ are defined recursively by (3.2) (with (2.17) and (3.3)), and the error terms $\tilde{\varepsilon}_{N,1}(n,x,\xi)$ and $\tilde{\varepsilon}^{\pm}_{N,2}(n,x,\xi)$ satisfy the bounds (3.4) and (3.6), respectively.

Subcase Ia. The following asymptotic expansion is uniformly valid for $-\infty < x < -n-1$ and $-\{1-\delta\}\, s_0(2-\beta)(n-x) \leqslant a < \infty$, where $s_0(\beta)$ is implicitly defined by (1.33) for $0 \leqslant \beta \leqslant 1$,

$$C_n^{(a)}(x) = \tilde{K}^+_{N,2}(n, x)\, e^{a/2} a^{(n-x)/2} \times \{a^2 - 2(n+x+1)\, a + (n-x)^2\}^{-1/4}\, \tilde{W}^+_{N,2}(n, x, \xi_a), \tag{8.3}$$

with

(8.4)

$$\tilde{K}^+_{N,2}(n, x) = (-1)^n \frac{|2x+1|^{(2x+1)/2}\,(2n+1)^{(2n+1)/2}}{\{2e(n-x)\}^{(n+x+1)/2}} \left[1 + \sum_{j=1}^{N-1} (-1)^j \frac{\tilde{a}_j^+(\beta)}{(n-x)^j}\right]^{-1},$$

and where the coefficients $\tilde{a}_j^+(\beta)$ are those that appear in the formal expansion

$$\frac{\Gamma(\frac{1}{2}\{\beta U + 1\})\, U^{(2U-\beta U-1)/2}}{\Gamma(U)\, e^U} \left(\frac{2e}{\beta}\right)^{\beta U/2} \sim 1 + \sum_{j=1}^{\infty} (-1)^j \frac{\tilde{a}_j^+(\beta)}{U^j}. \tag{8.5}$$

Subcase Ib. The following asymptotic expansion is uniformly valid for $-n-1 \leqslant x < -\frac{1}{2}$ and $-\infty < a \leqslant \{1-\delta\}\, s_0(\beta)(n-x)$

$$C_n^{(a)}(x) = n!\, e^{a/2}(-a)^{(n-x)/2}\, \{a^2 - 2(n+x+1) + (n-x)^2\}^{-1/4} \times\left[e^{(n-x)\pi i} \tilde{K}(n, x)\, \tilde{W}_{N,1}(n, x, \xi_b) + \frac{(-1)^n}{\Gamma(-x)} \tilde{K}^-_{N,2}(n, x)\, \tilde{W}^-_{N,2}(n, x, \xi_b)\right], \tag{8.6}$$

where

$$\tilde{K}(n, x) = \left(\frac{2}{e}\right)^{(n-x)/2} \frac{(n-x)^{(3n-x+2)/2}}{\Gamma(n-x+1)(2n+1)^{(2n+1)/2}}, \tag{8.7}$$

and

$$\tilde{K}^-_{N,2}(n, x) = \left(\frac{2e}{n-x}\right)^{(n+x+1)/2} \left[1 + \sum_{j=1}^{N-1} (-1)^j \frac{\tilde{a}_j^+(2-\beta)}{(n-x)^j}\right]^{-1}. \tag{8.8}$$

Subcase Ic. The expansion (8.6) is also uniformly valid for $-\frac{1}{2} \leqslant x \leqslant \delta^4 u - \frac{1}{2}$, but this time for $-\infty < a \leqslant \{1-\delta\}\, s^-(\beta)(n-x)$, where $s^-(\beta) = 1 - \beta - \sqrt{\beta^2 - 2\beta}$.

Case II. This case provides asymptotic expansions of $\delta^4 u - \frac{1}{2} \leqslant x \leqslant (1+\Delta)\, n + \frac{1}{2}\Delta$. Let $t = a/u$ and $\alpha(x) = (x-n)/u$. Then we introduce a new variable ζ by (4.14)–(4.17). Asymptotic expansions, in terms of Bessel functions, are given by

$$W_{2N+1,\,p}(u, \alpha, \zeta) = \zeta^{1/2} \mathscr{C}^{(p)}_{|x-n|}\left(\left(n+\frac{1}{2}\right)\zeta^{1/2}\right) \sum_{j=0}^{N} \frac{A_j(\alpha, \zeta)}{(n+\frac{1}{2})^{2j}} + \frac{\zeta}{n+\frac{1}{2}} \mathscr{C}^{(p)\prime}_{|x-n|}\left(\left(n+\frac{1}{2}\right)\zeta^{1/2}\right) \sum_{j=0}^{N-1} \frac{B_j(\alpha, \zeta)}{(n+\frac{1}{2})^{2j}} + \varepsilon_{2N+1,\,p}(u, \alpha, \zeta) \qquad (p = 1, 2, 3, 4), \tag{8.9}$$

where $\mathscr{C}^{(1)}_v(x) = J_v(x)$, $\mathscr{C}^{(2)}_v(x) = Y_v(x)$, $\mathscr{C}^{(3)}_v(x) = I_v(x)$, $\mathscr{C}^{(4)}_v(x) = K_v(x)$. The error terms $\varepsilon_{2N+1,1}(u, \alpha, \zeta)$, $\varepsilon_{2N+1,3}(u, \alpha, \zeta)$, and $\varepsilon_{2N+1,4}(u, \alpha, \zeta)$ satisfy the bounds (3.8), (3.14) and (3.15), respectively, of [1]. The error term $\varepsilon_{N+1,2}(u, \alpha, \zeta) = \operatorname{Im} \varepsilon^{(1)}_{2N+1}(u, \alpha, \zeta)$, where $\varepsilon^{(1)}_{2N+1}(u, \alpha, \zeta)$ satisfies the bound [1, Eq. (5.16)].

The coefficients $A_j(\alpha, \zeta)$ and $B_j(\alpha, \zeta)$ are defined via (4.13), (5.2) and (5.3), and we take the integration constants in (5.3) so that $A_j(\alpha, 0) = 0$ for $j = 1, 2, 3\ldots$.

Subcase IIa. The following asymptotic expansion is uniformly valid for $\delta^4 u - \frac{1}{2} \leqslant x < n$ and $0 < a \leqslant \{t^+(\alpha) - \delta\}\, u$, where $t^+(\alpha) = 2 + \alpha + 2\sqrt{1+\alpha}$:

$$C^{(a)}_n(x) = n!\, e^{a/2} a^{(n-x)/2} \left(\frac{\alpha^2 - \zeta}{(x-n)^2 + a^2 - 2(x+n+1)\, a}\right)^{1/4} \zeta^{-1/2} \times \left[\cos\{(n-x)\,\pi\} \mathbf{K}_{2N+1,1}(n, x)\, W_{2N+1,1}(u, \alpha, \zeta) - \frac{\Gamma(x+1)\sin\{(n-x)\,\pi\}}{\pi} K^{(1)}_{2N+1}(n, x)\, W_{2N+1,2}(u, \alpha, \zeta)\right], \tag{8.10}$$

where

$$\mathbf{K}_{2N+1,1}(n, x) = \sqrt{n+\frac{1}{2}}\, (1+\alpha)^{x/2+1/4} \left(\frac{2e}{2n+1}\right)^{(n-x)/2} \times \left[1 + (n-x) \sum_{j=0}^{N-1} \frac{B_j(\alpha, 0)}{(n+\frac{1}{2})^{2j+2}}\right]^{-1}, \tag{8.11}$$

and

$$K^{(1)}_{2N+1}(n, x) = \frac{\pi\sqrt{n+\frac{1}{2}}}{n!(1+\alpha)^{x/2+1/4}} \left(\frac{2n+1}{2e}\right)^{(n-x)/2} \times \left[1 - (n-x) \sum_{j=0}^{N-1} \frac{B_j(\alpha, 0)}{(n+\frac{1}{2})^{2j+2}} + \delta^{(1)}_{2N+1}(u, \alpha)\right]^{-1}. \tag{8.12}$$

In (8.12) the term $\delta^{(1)}_{2N+1}(u, \alpha)$ is given by (5.12), and is $O(u^{-2N-1})$. The coefficients $B_j(\alpha, 0)$ ($j=0, 1, 2, \ldots$) can be calculated in turn via the formal identity

$$\frac{\Gamma(u+\frac{1}{2})}{\Gamma(u+u\alpha+\frac{1}{2})}\left(\frac{u+u\alpha}{e}\right)^{u\alpha}(1+\alpha)^{u} \sim \left[1-\alpha\sum_{j=0}^{\infty}\frac{B_j(\alpha, 0)}{u^{2j+1}}\right]\left[1+\alpha\sum_{j=0}^{\infty}\frac{B_j(\alpha, 0)}{u^{2j+1}}\right]^{-1}. \tag{8.13}$$

The following asymptotic expansion is uniformly valid for $\delta^4 u-\frac{1}{2}\leqslant x<n$ and $-\infty<a<0$

$$C_n^{(a)}(x)=n!\,e^{a/2}\,|a|^{(n-x)/2}\left(\frac{\alpha^2-\zeta}{(x-n)^2+a^2-2(x+n+1)\,a}\right)^{1/4}|\zeta|^{-1/2} \times\left[\mathbf{K}_{2N+1,3}(n, x)\,W_{2N+1,3}(u, \alpha, \zeta) +\frac{(-1)^n}{\Gamma(-x)}K_{2N+1,4}(n, x)\,W_{2N+1,4}(u, \alpha, \zeta)\right], \tag{8.14}$$

where $\mathbf{K}_{2N+1,3}(n, x)=\mathbf{K}_{2N+1,1}(n, x)$ and

$$K_{2N+1,4}(n, x)=\frac{2\sqrt{n+\frac{1}{2}}}{n!(1+\alpha)^{x/2+1/4}}\left(\frac{2n+1}{2e}\right)^{(n-x)/2} \times\left[1-(n-x)\sum_{j=0}^{N-1}\frac{B_j(\alpha, 0)}{(n+\frac{1}{2})^{2j+2}}+\delta_{2N+1,4}(u, \alpha)\right]^{-1}. \tag{8.15}$$

In (8.15) the term $\delta_{2N+1,4}(u, \alpha)$ is given by (5.8), and is $O(u^{-2N-1})$.

Subcase IIb. The following asymptotic expansions are uniformly valid for $n\leqslant x\leqslant(1+\Delta)\,n+\frac{1}{2}\Delta$ and $-\infty<a\leqslant\{t^{+}(\alpha)-\delta\}\,u$

$$C_n^{(a)}(x)=K_{2N+1,1}(n, x)\,e^{a/2}\,|a|^{-(x-n)/2} \times\left(\frac{\alpha^2-\zeta}{(x-n)^2+a^2-2(x+n+1)\,a}\right)^{1/4}|\zeta|^{-1/2}\,W_{2N+1,p}(u, \alpha, \zeta), \tag{8.16}$$

with $p=1$ for $a>0$ and $p=3$ for $a<0$. Here

$$K_{2N+1,1}(n,x)=\frac{\sqrt{n+\frac{1}{2}}\;\Gamma(x+1)}{(1+\alpha)^{x/2+1/4}}\left(\frac{2e}{2n+1}\right)^{(x-n)/2}\times\left[1+(x-n)\sum_{j=0}^{N-1}\frac{B_j(\alpha,0)}{(n+\frac{1}{2})^{2j+2}}\right]^{-1}. \tag{8.17}$$

Case III. This case provides asymptotic expansions for $(1+\Delta)\,n+\frac{1}{2}\Delta\leqslant x<\infty$. Let $r=a/(x+\frac{1}{2})$ and $v(x)=\sqrt{(n+\frac{1}{2})/(x+\frac{1}{2})}$. Then define a Liouville variable η by (6.10).

The following asymptotic expansion is uniformly valid for $(1+\Delta)\,n+\frac{1}{2}\Delta\leqslant x<\infty$ and $-\infty<a\leqslant\{1-\delta\}(1-v(x))^2\,(x+\frac{1}{2})$

$$C_n^{(a)}(x)=\hat{K}(n,x)\,e^{a/2}a^{-(x-n)/2}\{(x-n)^2+a^2-2(x+n+1)\,a\}^{-1/4}\times\left\{e^{(x+1/2)\,\eta}\left[1+\sum_{j=1}^{N-1}\frac{\hat{A}_j(\eta)}{(x+\frac{1}{2})^j}\right]+\hat{\varepsilon}_{N,1}(x,\eta)\right\}, \tag{8.18}$$

where $\hat{A}_j(\eta)$ $(j=1,2,3,\ldots)$ are defined recursively by (6.15) and (7.2), $\hat{\varepsilon}_{N,1}(x,\eta)$ is bounded by (7.4), and

$$\hat{K}(n,x)=\frac{\Gamma(x+1)(n+\frac{1}{2})^{(x+n+1)/2}\,(x-n)^{x-n+1/2}}{\Gamma(x-n+1)\,e^{(x-n)/2}}\left(\frac{2}{x+\frac{1}{2}}\right)^{x+1/2}. \tag{8.19}$$

9. NUMERICAL CALCULATIONS

In Table 2 exact and approximate values of $C_n^{(a)}(x)$ are given, for $n=30$, $a=10$, and various values of x. For simplicity just one term $(N=0)$ is taken for each of the asymptotic expansions. Thus the relative errors are $O\{(n-x)^{-1}\}$, $O\{n^{-1}\}$ and $O\{x^{-1}\}$, for Cases I, II and III, respectively.

The apparent discrepancy between the exact value and the Subcase Ic approximation when $x=1$ is due to this x value being close to a zero of $C_{30}^{(10)}(x)$ (which occurs at $x=1.000118\ldots$), coupled with the very large amplitude of oscillation of $C_{30}^{(10)}(x)$ in the vicinity of $x=1$. Subcase Ic does however break down near $x=5.5$. This is because $s^-(\beta)(n-x)=9.4445\ldots$ for $x=5.5$ and $n=30$, and this is close to the fixed

TABLE 2

x	$C_{30}^{(10)}(x)$	**Subcase Ia** **Eq. = (8.3)**	
-1000000	1.0007353E+180	1.0007351E+180	
-10000	1.0761132E+120	1.0760998E+120	
-1000	2.0564092E+90	2.0561661E+90	
-100	5.0967674E+62	5.0921906E+62	**Subcase Ib,c**
-50	5.6576204E+55	5.6484443E+55	**Eq. (8.6)**
-31	3.7697641E+51	3.7603029E+51	3.7603E+51
-20	1.7686791E+48		1.7633E+48
-10	9.5270670E+43		94831E+43
-5	6.9328090E+40	**Subcase IIa**	6.8832E+40
-1	5.8425826E+36	**Eq. (810)**	5.7600E+36
-0.5	7.3995373E+35	7.3760E+35	7.2804E+35
0	1.0000000E+30	1.0022E+30	1.0118E+30
1	-2.0000000E+30	-2.0039E+30	2.0337E+30
1.5	1.7756442E+33	1.7714E+33	1.7107E+33
5.5	1.5625004E+31	1.5605E+31	5.9718E+30
10	-8.3780768E+30	-8.3866E+30	
20.5	-6.6880619E+31	-6.6856E+31	**Subcase IIb**
25	-3.8389361E+32	-3.8377E+32	**Eq. (8.16)**
30	-3.3731664E+33	-3.3696E+33	-3.3696E+33
40	-2.9453149E+36		-2.9459E+36
50	5.8124448E+39		5.8141E+39
60	1.6708355E+43		1.6704E+43
70	4.7094723E+47		4.7159E+47
80	1.7733979E+51		1.7738E+51
90	6.0595887E+53		6.0605E+53
100	6.8477444E+55	**Case III**	6.8486E+55
110	3.8030198E+57	**Eq. (8.18)**	3.8034E+57
121.5	2.0731825E+59	2.0766321E+59	2.0733E+59
130	2.8767120E+60	2.8802952E+60	
140	4.7973025E+61	4.8017594E+61	
150	6.2327099E+62	6.2371969E+62	
200	1.7648248E+67	1.7653230E+67	
300	1.4835505E+73	1.4836876E+73	
1000	4.7238924E+89	4.7239196E+89	
10000	9.2900121E+119	9.2900126E+119	
1000000	9.9926526E+179	9926526E+179	

value $a = 10$ under consideration; recall that Subcase Ic is valid for $-\infty < a \leqslant \{1-\delta\}\, s^{-}(\beta)(n-x)$. It is interesting to note that the Subcase IIa approximation remains good even for small values of x: this is because we have only taken one term in our expansion, and hence none of the coefficients $B_j(\alpha, 0)$ appear (which become unbounded as $\alpha(x) \to -1$, i.e., for $n \to \infty$ and bounded x).

10. APPENDIX

We prove the montonicity of $s_0(\beta)$ by the following theorem.

THEOREM A.1. *The function* $s_0(\beta)$, *which is implicitly defined by* (1.33), *satisfies* $\mathrm{d}s_0/\mathrm{d}\beta \leqslant 0$ *for* $0 \leqslant \beta \leqslant 1$.

If we differentiate (1.33) with respect to β, and solve for $\mathrm{d}s_0/\mathrm{d}\beta$, we find after some simplification that

$$\frac{\mathrm{d}s_0}{\mathrm{d}\beta} = -\frac{s_0\{\ln(\beta(2-\beta)) - 2\ln(1-\beta-s_0+S_0)\}}{2S_0}, \tag{A.1}$$

where $S_0(\beta)$ is defined by (1.34). Therefore the theorem follows immediately from the following result.

LEMMA A.2.

$$\ln(\beta(2-\beta)) - 2\ln(1-\beta-s_0+S_0) \geqslant 0, \tag{A.2}$$

for $0 < \beta \leqslant 1$, *where* $s_0(\beta) \in (0, 1]$ *satisfies* (1.33).

Proof. The complication in proving this is that $s_0(\beta)$ is not explicitly given. We cannot prove the result by simply replacing $s_0(\beta)$ by an arbitrary $s \in (0, 1]$, since the left hand side of the inequality is negative if $0 < s_0(\beta) < 1-\beta$ (which *a posteriori* is false). Instead, we first appeal to (1.33) to obtain the alternative representation

$$\ln(\beta(2-\beta)) - 2\ln(1-\beta-s_0+S_0) = \frac{2}{\beta} L(\beta, s_0), \tag{A.3}$$

where

$$L(\beta, s) = \ln\left(\frac{s(2-\beta)}{s^2+\beta s+1-(s+1)\,S(s)}\right) - S(s), \tag{A.4}$$

in which

$$S(s) = \sqrt{s^2-2s+2\beta s+1}\,. \tag{A.5}$$

We shall show that $L(\beta, s)$ is nonnegative for all $s \in (0, 1]$. Now, for fixed β, differentiation with respect to s leads to

$$\frac{\partial}{\partial s} L(\beta, s) = -\frac{(1-s)^2+\beta s}{sS(s)}, \tag{A.6}$$

which is clearly negative for $0<\beta\leqslant 1$ and $0<s\leqslant 1$. Hence for these ranges of β and s

$$L(\beta, s)\geqslant L(\beta, 1)=\ln\left(\frac{2+\sqrt{2\beta}}{2-\sqrt{2\beta}}\right)-\sqrt{2\beta}\,. \tag{A.7}$$

To show that the right hand side of (A.7) is positive, we first observe that it is increasing for $0<\beta\leqslant 1$:

$$\frac{d}{d\beta}\left\{\ln\left(\frac{2+\sqrt{2\beta}}{2-\sqrt{2\beta}}\right)-\sqrt{2\beta}\right\}=\frac{\sqrt{\beta}}{\sqrt{2}\,(2-\beta)}>0. \tag{A.8}$$

Thus

$$L(\beta, s)\geqslant L(\beta, 1)\geqslant L(0, 1)=0, \tag{A.9}$$

for $0<\beta\leqslant 1$ and $0<s\leqslant 1$. Lemma A2 now follows from (A.3) and (A.9).

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